Auctions: Theory and Applications

Contents

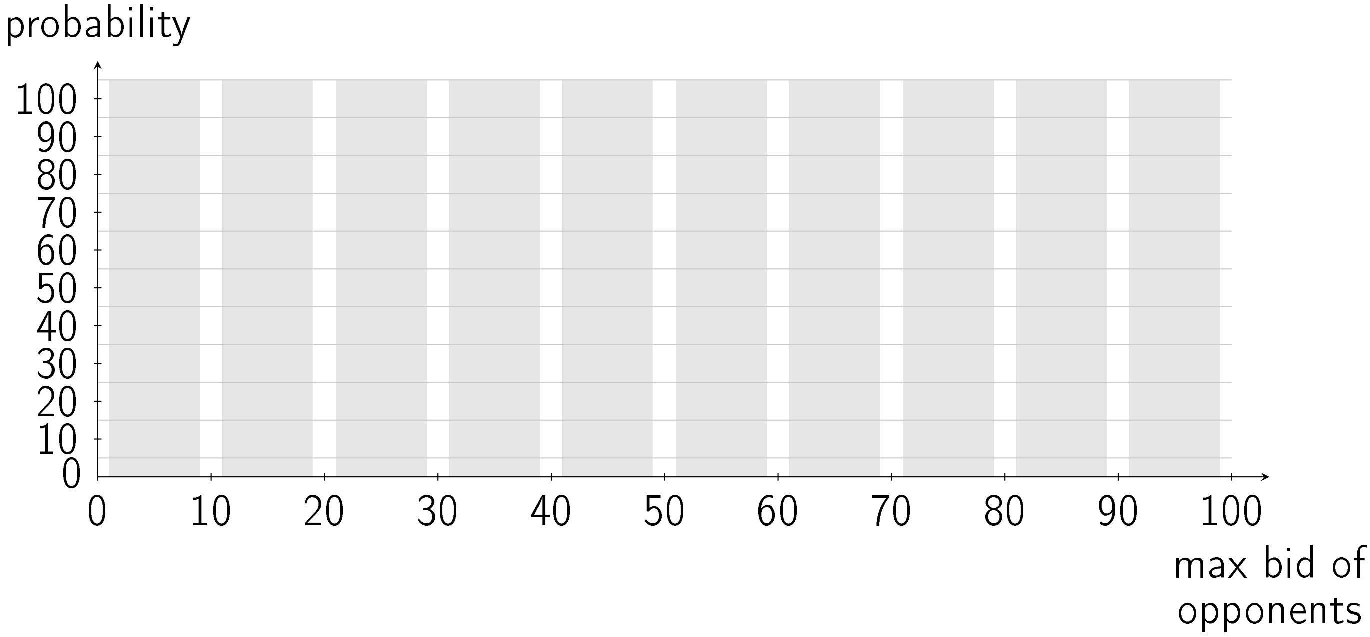
Chapter 1

An Experiment

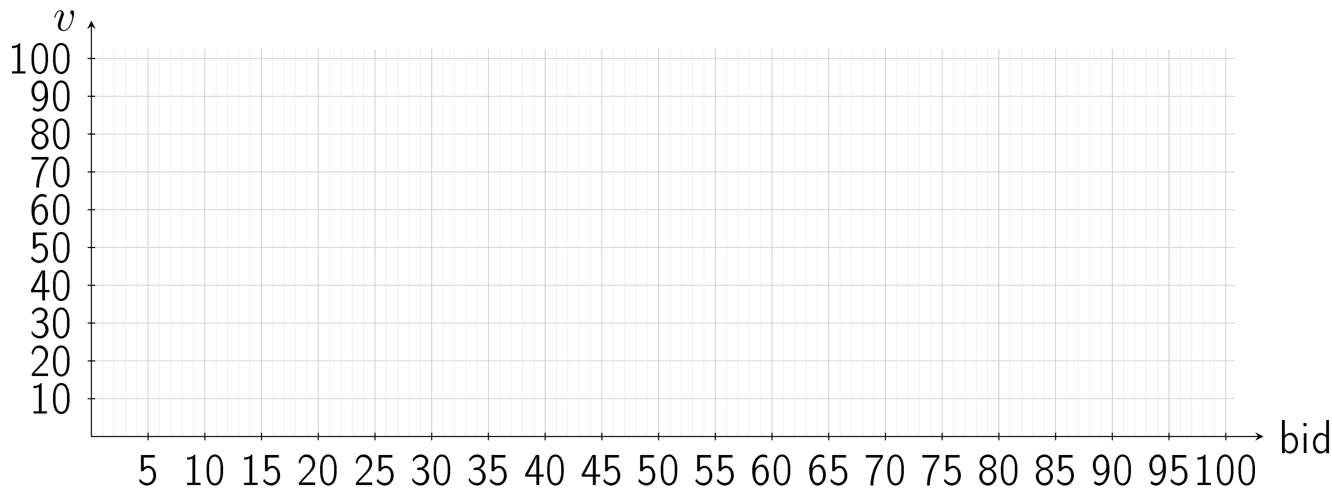
1.1 A First Price Sealed Bid Auction With Private Values

- There is one object for sale.
- The object is of value €v_i to bidder $i = 1, \ldots, n$.
- v is private information to bidder i.
- Bidder i knows that v_j is uniform iid on $\{10, 20, \ldots, 100\}$, but i does not know v_j.
- Each bidder submits a bid without knowing the bids of the other bidders.
- The bidder with the highest bid wins the object and pays the own bid.
- The payoff of the winner is own valuation - payment. All other bidders receive zero.

With which probability will the highest bid of your opponents be in any of the intervals?



Mark your own bids for any of your potential valuations in the following graph:



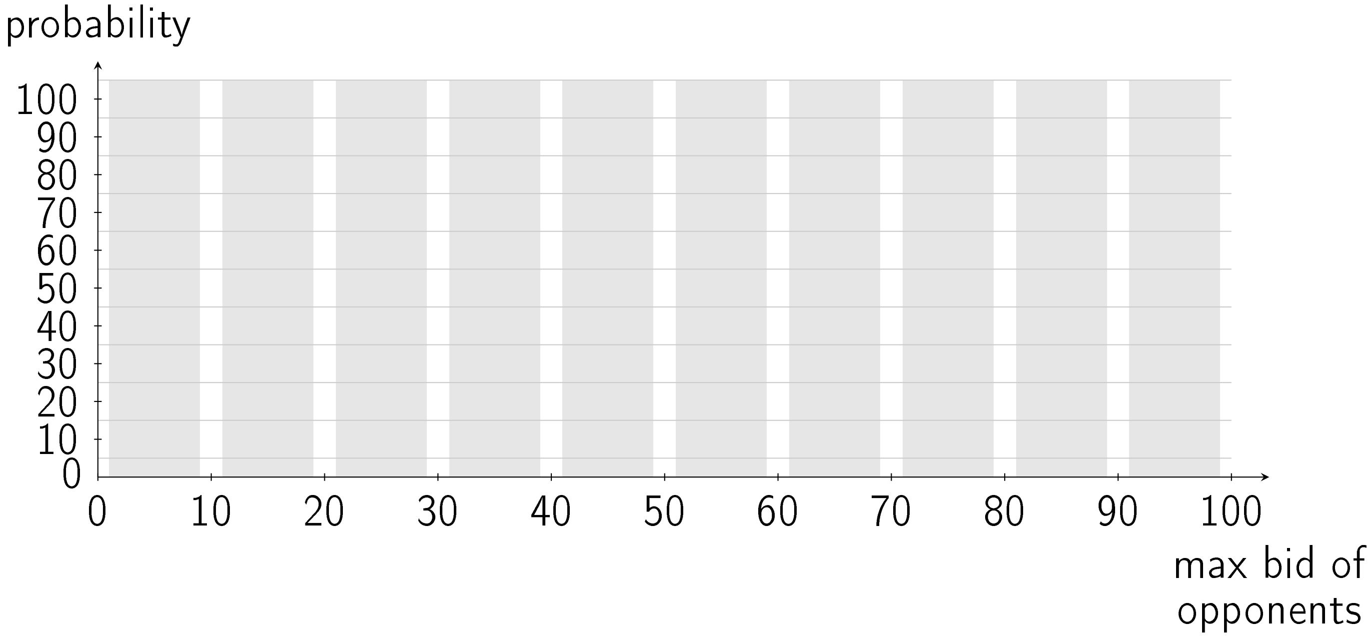
1st-price, private info name: ____________

1.2 2nd Price Sealed Bid Auction With Private Values

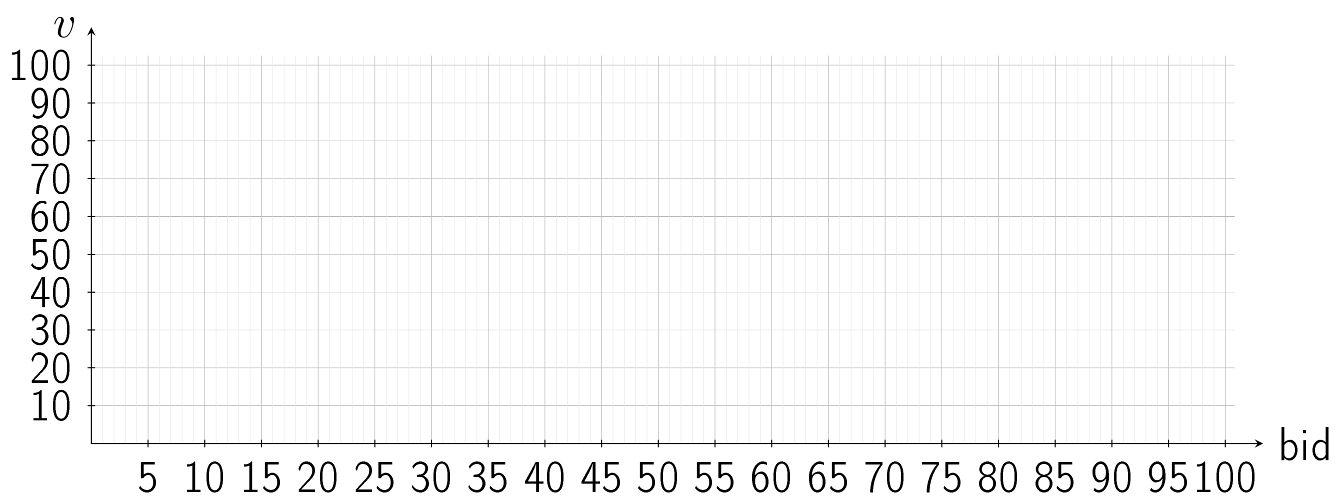
The rules of the auction are exactly the same as in the previous auction with the exception of what the winner has to pay:

- The bidder with the highest bid wins the object and pays the second highest bid.

With which probability will the highest bid of your opponents be in any of the intervals?



Mark the bids for any of your potential valuations in the following graph:



2nd-price, private info name: ____________

Chapter 2

A Survey on Auction Theory

Taken from Paul Klemperer 'Auctions: Theory and Practice' (2004), chapter one.

2.1 Four Standard Auction Types

Focus on single object auctions.

- **ascending-bid auction (aka open, oral, or English auction)**
  Price rises continuously, bidders gradually quit the auction. Bidders observe when their competitors quit and once someone quits, she is not let back in. Last bidder who remains wins the object at the final price.

- **descending-bid auction (aka Dutch auction)**
  The auctioneer starts at very high price, and then lowers the price continuously. The first bidder who accepts the price wins the object at that price.

- **first-price sealed bid auction**
  Each bidder independently submits a single bid without observing oth-

ers' bids. The object is sold to the bidder who makes the highest bid at the price of the highest bid.

- **second-price sealed bid auction (aka Vickrey auction)**
  Each bidder independently submits a single bid without observing others' bids. The object is sold to the bidder who makes the highest bid at the price of the second highest bid.

2.2 Basic Model of Auctions

Focus on symmetric and risk neutral bidders.

Private-Value Model: each bidder knows how much she values the object for sale, but her value is private information to herself. Each bidder knows the overall distribution of values among bidders.

Common-Value Model: the actual value of the object is the same for everyone, but bidders may have different information about what the value is.

General Model: each bidder receives a private information signal. Each bidder's value is a function of all the signals.

Bidders: $i = 1, \ldots, n$
The value of the object to bidder i: v_i
The private signal of bidder i: t_i

Private-Value Model:

$$v_i = v_i(t_i) \text{ for each } i = 1, \ldots, n$$

<u>Common-Value Model:</u>

$$v_i = v(t_1, \dots, t_n) \text{ for all } t_1, \dots, t_n \text{ and } i = 1, \dots, n$$

<u>General Model:</u>

$$v_i = v_i(t_1, \dots, t_n),\ i = 1, \dots, n$$

Set of admissable valuations: $\mathcal{V}_i$. Set of admissable signals: $\mathcal{T}_i$.

Example: Oilfields

Consider an oilfield with an unkown capacity of v million gallons. v equals 15.000 or 1.500 each with probability $\frac{1}{2}$. An experts sends the signal $t = H$ with prob. $\frac{2}{3}$ and $t = L$ with prob $\frac{1}{3}$, if $v = 15.000$. If $v = 1.500$, then the expert sends $t = H$ with prob $\frac{1}{3}$ and $t = L$ with prob $\frac{2}{3}$. Hence

$$Prob(v = 15.000|t = H) = \frac{\frac{1}{2}\cdot\frac{2}{3}}{\frac{1}{2}\cdot\frac{2}{3}+\frac{1}{2}\cdot\frac{1}{3}} = \frac{2}{3}$$

and

$$Prob(v = 15.000|t = L) = \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{2}{3}} = \frac{1}{3}\ .$$

Given the signal, the expected capacity of the oilfield is

$$v(t) := E[v|t] = \begin{cases} \frac{2}{3}\cdot 15.000 + \frac{1}{3}\cdot 1.500 = 10.500 & \text{if } t = H \\ \frac{1}{3}\cdot 15.000 + \frac{2}{3}\cdot 1.500 = 6.000 & \text{if } t = L \end{cases}$$

Consider now two signals (experts) t_1, t_2:

We have

$$\begin{aligned} Prob(15.000|H,H) &= \frac{\frac{1}{2}\cdot\frac{4}{9}}{\frac{1}{2}\cdot\frac{4}{9}+\frac{1}{2}\cdot\frac{1}{9}} = \frac{4}{5} \\ Prob(15.000|H,L) &= \frac{\frac{1}{2}\cdot\frac{2}{9}}{\frac{1}{2}\cdot\frac{2}{9}+\frac{1}{2}\cdot\frac{2}{9}} = \frac{1}{2} = Prob(15.000|L,H) \\ Prob(15.000|L,L) &= \frac{\frac{1}{2}\cdot\frac{1}{9}}{\frac{1}{2}\cdot\frac{4}{9}+\frac{1}{2}\cdot\frac{1}{9}} = \frac{1}{5} \end{aligned}$$

Hence

$$\underbrace{E[v|t_1,t_2]}_{v(t_1,t_2):=} = \begin{cases} \frac{4}{5}\cdot 15.000 + \frac{1}{5}\cdot 1.500 = 12.300 & \text{if } (t_1,t_2) = (H,H) \\ \frac{1}{2}\cdot 15.000 + \frac{1}{2}\cdot 1.500 = 8.250 & \text{if } (t_1,t_2) \in \{(H,L),(L,H)\} \\ \frac{1}{5}\cdot 15.000 + \frac{4}{5}\cdot 1.500 = 4.200 & \text{if } (t_1,t_2) = (L,L) \end{cases}$$

2.3 Bidding in the Standard Auctions

Descending Auction

The auction is dynamic, but the decision problem is static:

Each bidder i chooses a price b_i at which she will call out, given t_i.

→ strategy: $b_i : \mathcal{T}_i \to \mathbb{R}_+$.

The bidder with the highest bid wins the object and pays the highest bid.

First-Price Sealed-Bid Auction

Each bidder must choose a bid b_i given t_i.

→ strategy: $b_i : \mathcal{T}_i \to \mathbb{R}_+$.

The bidder with the highest bid wins the object and pays the highest bid.

⇒ the descending auction and the 1st-p s-b auction are strategically equivalent. (Strategies are the same, payoffs are the same).

Second-Price Sealed-Bid Auction

Each bidder must choose a bid b_i, given t_i.
→ strategy: $b_i : \mathcal{T}_i \to \mathbb{R}_+$.
The bidder with the highest bid wins and pays the second highest bid.

Ascending Auction with Private Values: $v_i = v_i(t_i)$.

If i observes that j drops out, i learns something about v_j and hereby about t_j. t_j, however is irrelevant for v_i.

Decision problem: at which price will I drop out, given my signal?
→ strategy: $b_i : \mathcal{T}_i \to \mathbb{R}_+$.
The bidder with the highest value wins and pays the second highest value.

The ascending auction and the 2nd-p s-b auction are strategically equivalent, if values are private.

Ascending Auction with Common Values: $v_i = v(t_1, \ldots, t_n)$

If some other bidder j drops out, i learns something about the value of the object.
Decision problem: at which price will I drop out, given my signal and the drop outs of my competitors?
The strategy space is richer than in the private value model.

Winner's Curse

If the signals are symmetric around the true value and if bids are monotonic in the signals, the winning bid is triggered by a signal which exceeds the value.

Terminology

Descending & 1st-p s-b auction are equivalent for single object auctions. We subsume these two types as 'first-price auctions'.

Ascending & 2nd-p s-b auctions are equivalent under private values. We subsume these two types as 'second-price auctions'.

2.4 Analysis (a primer)

Second-price auction with independent private values

Strategy for player $i = 1, \ldots, n$: $\beta_i : \mathcal{V}_i \to \mathbb{R}_+$

DEFINITION:
$\tilde{\beta}_i$ is weakly dominated by β_i if

$$E_{v_{-i}} u_i(\tilde{\beta}_i(v_i), \beta_{-i}|v_i) \leq E_{v_{-i}} u_i(\beta_i(v_i), \beta_{-i}|v_i) \;\forall\; v_i \in \mathcal{V}_i, \beta_{-i} \in S_{-i}$$

with a strict inequality for at least one $\beta_{-i} \in S_{-i}$.

DEFINITION:
β_i is weakly dominant if any $\tilde{\beta}_i \in S_i$, $\tilde{\beta}_i \neq \beta_i$ is weakly dominated by β_i.

DEFINITION:
The strategy β_i with $\beta_i(v_i) = v_i \;\forall\; v_i \in \mathcal{V}_i$ is called 'bid your value'.

PROPOSITION:

Bid your value is a weakly dominant strategy.

Denote by $\hat{b}_{-i}$ the highest bid from i's opponents: $\hat{b}_{-i} = \max_{j \neq i} b_j$.

Payoff function for given bids b_i, b_{-i}:

$$u_i(b_i, b_j|v_i) = \begin{cases} v_i - b_j & \text{if } b_i > \hat{b}_{-i} \\ \frac{v_i - b_i}{|\{j : b_j = b_i\}|} & \text{if } b_i = \hat{b}_{-i} \\ 0 & \text{if } b_i < \hat{b}_{-i} \end{cases}$$

PROOF:

$$u_i(b_i = v_i, b_j|v_i) = \begin{cases} v_i - b_j & \text{if } v_i > \hat{b}_{-i} \\ 0 & \text{if } v_i \leq \hat{b}_{-i} \end{cases}$$

Consider now some valuation $v_i \in \mathcal{V}_i$ and two strategies $\hat{\beta}_i$ and $\check{\beta}_i$ with $\hat{\beta}_i(v_i) = \hat{v}_i > v_i > \check{v}_i = \check{\beta}_i(v_i)$.

	$\hat{b}_{-i} > \hat{v}_i$	$\hat{b}_{-i} = \hat{v}_i$	$v_i < \hat{b}_{-i} < \hat{v}_i$	$\hat{b}_{-i} = v_i$
$u_i(b_i = \hat{v}_i, b_{-i}\|v_i)$	0	$\frac{v_i - \hat{v}_i}{\|\{j : b_j = \hat{v}_i\}\|}$	$v_i - \hat{b}_{-i}$	0
$u_i(b_i = v_i, b_{-i}\|v_i)$	0	0	0	0
$u_i(b_i = \check{v}_i, b_{-i}\|v_i)$	0	0	0	0

	$\check{v}_i < \hat{b}_{-i} < v_i$	$\hat{b}_{-i} = \check{v}_i$	$\hat{b}_{-i} < \check{v}_i$
$u_i(b_i = \hat{v}_i, b_{-i}\|v_i)$	$v_i - \hat{b}_{-i}$	$v_i - \check{v}_i$	$v_i - \hat{b}_{-i}$
$u_i(b_i = v_i, b_{-i}\|v_i)$	$v_i - \hat{b}_{-i}$	$v_i - \check{v}_i$	$v_i - \hat{b}_{-i}$
$u_i(b_i = \check{v}_i, b_{-i}\|v_i)$	0	$\frac{v_i - \check{v}_i}{\|\{j : b_j = \check{v}_i\}\|}$	$v_i - \hat{b}_{-i}$

Therefore, for any valuation $v_i \in \mathcal{V}_i$ of bidder i and strategy $\tilde{\beta}_i$ with $\tilde{\beta}_i(v_i) \neq v_i$ there exists some $(n-1)$-tupel of strategies β_{-i} with $\beta_j(v_j) \in (v_i, \tilde{\beta}_i(v_i)]$ for all $v_j \in \mathcal{V}_j$ and $j \neq i$, if $\tilde{\beta}_i(v_i) > v_i$ and with $\beta_j(v_j) \in [\tilde{\beta}_i(v_i), v_i)$ for all $v_j \in \mathcal{V}_j$ and $j \neq i$, if $\tilde{\beta}_i(v_i) < v_i$ such that

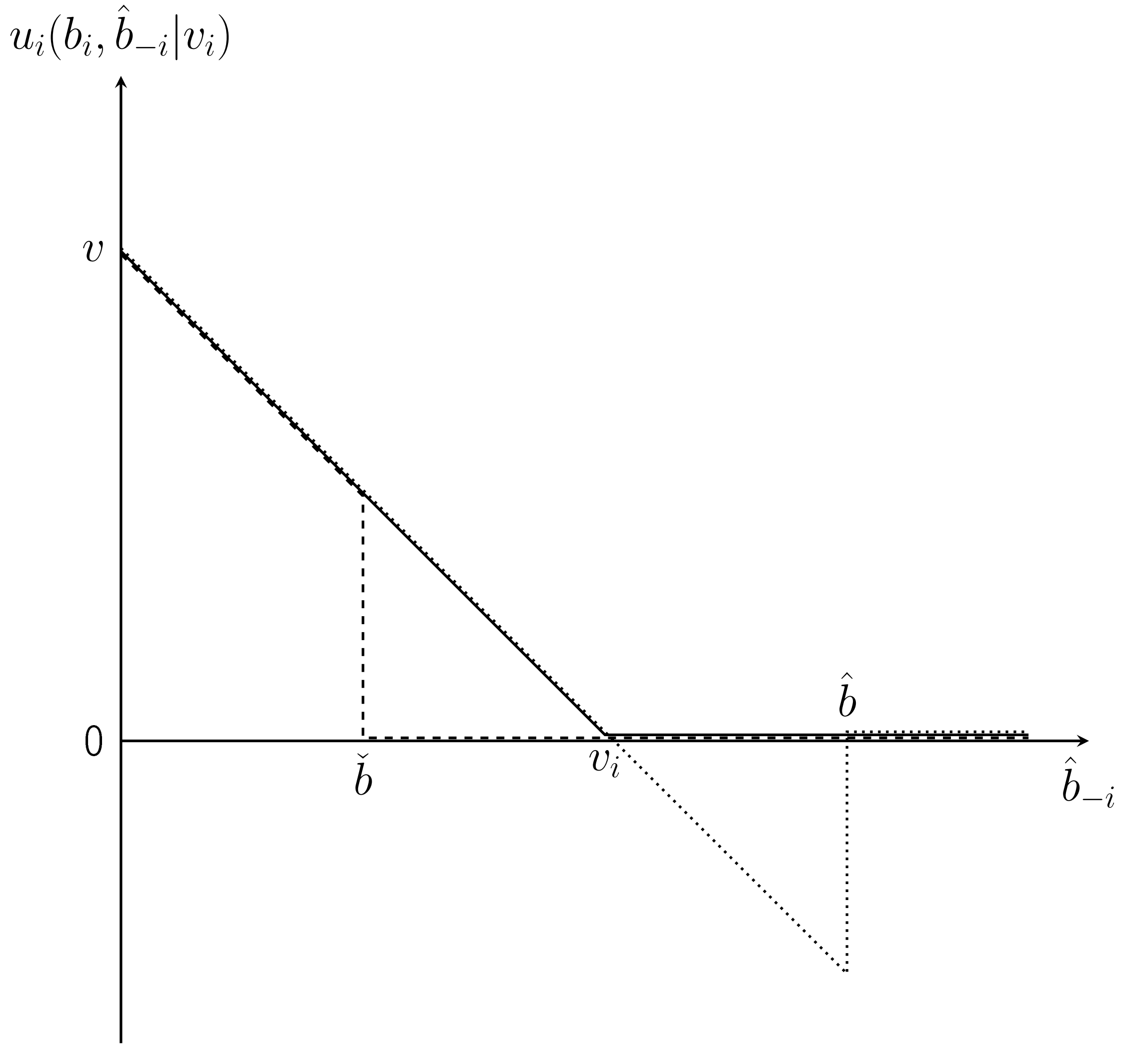
$$E_{v_{-i}} u_i(\beta_i(v_i), \beta_{-i}|v_i) > E_{v_{-i}} u_i(\tilde{\beta}_i(v_i), \beta_{-i}|v_i)$$

and with

$$E_{v_{-i}} u_i(\beta_i(v_i), \beta_{-i}|v_i) \geq E_{v_{-i}} u_i(\tilde{\beta}_i(v_i), \beta_{-i}|v_i)$$

for all other strategies β_{-i}. □

Suppose the maximal bid of the opponents is $\check{b} < v_i$ (dashed line) or that the maximal bid of the opponents is $\hat{b} > v_i$ (dotted line), where v_i is the bidder's valuation for the object.



The solid line represents the payoff of a bidder who drops out at the price equal to her valuation v. The solid line is never below the dashed or the dotted line, but strictly above the dashed or dotted line for some opponent's bid b_j.

Exercise:

Find another equilibrium (in pure strategies)!

First-price sealed-bid auction with independent private uniform values and two bidders

Two bidders $i = 1, 2$ with valuations $v_i \sim U[0, 1]$. Given the opponent's bid b_j, the own bid b_i and the own valuation v_i, bidder i's payoff is

$$u_i(b_i, b_j|v_i) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ \frac{1}{2}(v_i - b_i) & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

Suppose the players use strategies $\beta_i : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy

- strict monotonicity: $\beta_i(v_i) > \beta_i(\tilde{v}_i) \Leftrightarrow v_i > \tilde{v}_i$
- continuity
- differentiability
- symmetry $(\beta_i(v_i) = \beta_j(v_j) \Leftrightarrow v_i = v_j)$.

Then for each player i the inverse bid function exists and is given by $\beta_i^{-1}(b)$ such that $\beta_i^{-1}(\beta_i(v_i)) = v_i$. Given bid $b_i \in \mathbb{R}_+$ the probability that player i wins the auction is

$$Prob(b_i > \beta_j(v_j)) = \begin{cases} 0 & \text{if } b_i < \beta_j(0) \\ Prob(\beta_j^{-1}(b_i) > v_j) = \beta_j^{-1}(b_i) & \text{otherwise} \\ 1 & \text{if } b_i > \beta_j(1) \end{cases} .$$

If $\beta_j(0) \leq b_i \leq \beta_j(1)$, the expected payoff of player i is

$$(v_i - b_i) \cdot \beta_j^{-1}(b_i) \ .$$

Maximizing with respect to b_i gives the first order condition for $b_i = \beta_i(v_i)$

$$-\beta_j^{-1}(\beta_i(v_i)) + (v_i - \beta_i(v_i)) \cdot \left. \frac{\partial \beta_j^{-1}(b)}{\partial b} \right|_{b=\beta_i(v_i)} = 0 \ .$$

Symmetry: $\beta_i(v) = \beta_j(v) = \beta(v)$ such that

$$-\beta^{-1}(\beta(v)) + (v - \beta(v)) \cdot \left. \frac{\partial \beta^{-1}(b)}{\partial b} \right|_{b=\beta(v)} = 0 \ .$$

Clearly $\beta^{-1}(\beta(v)) = v$ and $\left. \frac{\partial \beta^{-1}(b)}{\partial b} \right|_{b=\beta(v)} = \frac{1}{\frac{\partial \beta(v)}{\partial v}}$, hence

$$-v + (v - \beta(v)) \cdot \frac{1}{\beta'(v)} = 0$$

or

$$v = \beta'(v) \cdot v + \beta(v) = (\beta(v) \cdot v)' \ .$$

Integrating LHS:

$$\int_0^{\hat{v}} v dv = \frac{1}{2} \cdot \hat{v}^2$$

Intergrating RHS:

$$\int_0^{\hat{v}} (\beta(v) \cdot v)' \, dv = \beta(\hat{v}) \cdot \hat{v}$$

$\Rightarrow \beta(\hat{v}) = \frac{1}{2} \cdot \hat{v}$.

Therefore there is a unique symmetric equilibrium with continuous and strictly monotonic strategies in which both bidders $i = 1, 2$ bid according to

$$\beta_i(v_i) = \frac{1}{2} \cdot v_i \ .$$

Expected Equilibrium Revenue

First-Price Auctions

$$E\left[\frac{1}{2}\max\{v_1,v_2\}\right] = \frac{1}{2}\int_0^1\int_0^{v_1} v_1 dv_2 + \int_{v_1}^1 v_2 dv_2 dv_1 = \frac{1}{4}\int_0^1 1+v_1^2 dv_2 = \frac{1}{3}$$

Second-price Auctions

Two bidders $i=1,2$ with valuations $v_i \sim U[0,1]$, $b_i(v_i)=v_i$

$$E[\min\{v_1,v_2\}] = \int_0^1\int_0^{v_1} v_2 dv_2 + \int_{v_1}^1 v_1 dv_2 = \int_0^1 \frac{1}{2}v_1^2 + v_1(1-v_1)dv_1 = \frac{1}{3}$$

What is the expected revenue of 'the other equlibrium'?

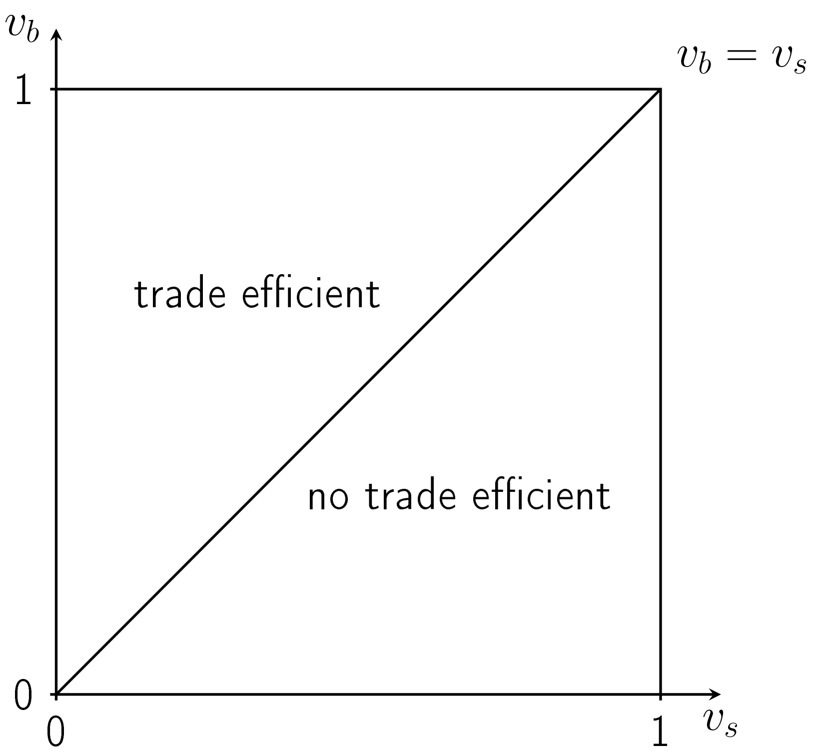
2.5 Double Auctions

Buyers and sellers are treated symmetrically. Buyers bid and sellers ask.
· Chatterjee and Samuelson (1983) 'Bargaining under incomplete information' *OR*
· Gibbons (1992): A primer in game theory p. 159

One buyer, one seller who holds one unit of a good.
Valuation of buyer: $v_b \sim U[0,1]$
Valuation of seller: $v_s \sim U[0,1]$

First Best

If $v_b > v_s$ a trade with any price $\in [v_s, v_b]$ is a Pareto improvement.



1st-best gains from trade

$$\int_0^1 \int_{v_s}^1 v_b - v_s dv_b dv_s = \frac{1}{6}$$

strategies

Strategies for buyer and seller: $\beta, \sigma : [0,1] \to \mathbb{R}_+$.

Standard auction

Suppose the seller *asks* $s \in \mathbb{R}_+$ and the bidder *bids* $b \in \mathbb{R}_+$.

- $s > b$: seller 'wins' (keeps the object).
- $s = b$: both win with prob. $\frac{1}{2}$.

- $s < b$: buyer wins.

1-st price auction: winner pays $\max\{s, b\}$ to seller.

2-nd price auction: winner pays $\min\{s, b\}$ to seller.

λ-price auction

Buyer bids $b \geq 0$ and seller asks $s \geq 0$.

Trade takes place if and only if $b \geq s$.

Price: $\lambda \cdot b + (1 - \lambda) \cdot s$, with $\lambda \in (0, 1)$.

Payoffs

$$u_s(s, b|v_s) = \begin{cases} \lambda \cdot b + (1 - \lambda) \cdot s & \text{, if } b \geq s \\ v_s & \text{, if } b < s \end{cases}$$

$$u_b(b, s|v_b) = \begin{cases} v_b - \lambda \cdot b - (1 - \lambda) \cdot s & \text{, if } b \geq s \\ 0 & \text{, if } b < s \end{cases}$$

Expected payoffs

$$\begin{aligned} E_{v_b} u_s(s, \beta(v_b)|v_s) &= prob\{s > \beta(v_b)\} \cdot v_s \\ &+ prob\{s \leq \beta(v_b)\} \cdot (\lambda \cdot E_{v_b}[\beta(v_b)|\beta(v_b) \geq s] + (1 - \lambda) \cdot s) \end{aligned}$$

$$E_{v_s} u_b(\sigma(v_s), b|v_b) = prob\{\sigma(v_s) > b\} \cdot 0$$
$$+prob\{\sigma(v_s) \leq b\} \cdot (v_b - (\lambda \cdot b + (1 - \lambda) \cdot E_{v_s}[\sigma(v_s)|b \geq \sigma(v_s)]))$$

Nash equilibrium

The strategies $(\beta,\ \sigma)$ are a Bayes Nash equilibrium, if $\forall\ v_b, v_s \in [0,1]$

$$\sigma(v_s) \in \underset{s\ \in\ \mathbb{R}_+}{\arg\max}\ Eu_s(s, \beta(v_b)|v_s)$$

and

$$\beta(v_b) \in \underset{b\ \in\ \mathbb{R}_+}{\arg\max}\ Eu_b(b, \sigma(v_s)|v_b)\ .$$

There are many Bayes Nash equilibria of this game!!

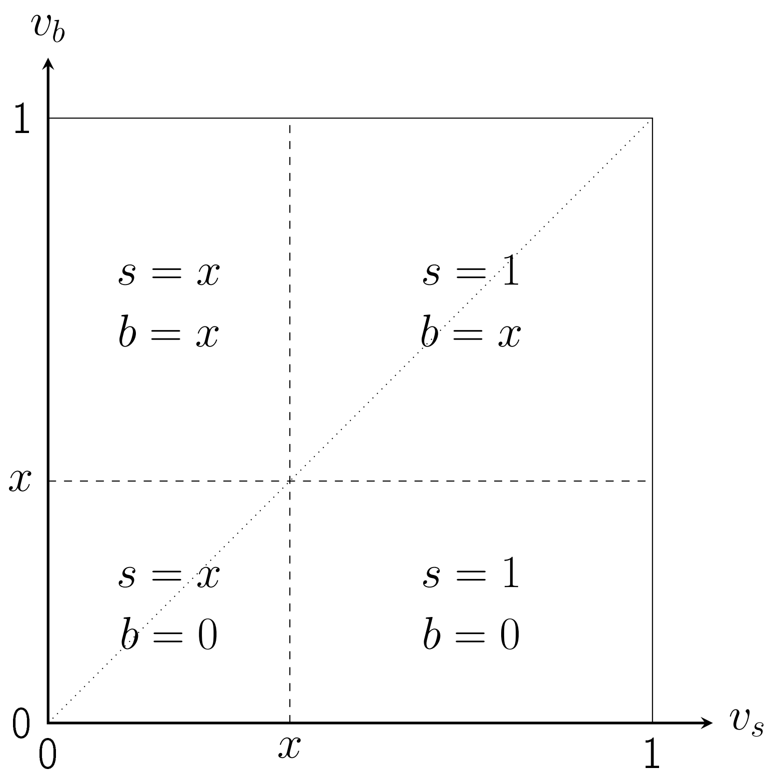
One-Price-Nash equilibrium

Buyer strategy:

$$\beta(v_b) = \begin{cases} 0 & \text{if } v_b < x \\ x & \text{if } v_b \geq x \end{cases}$$

Seller strategy:

$$\sigma(v_s) = \begin{cases} x & \text{if } v_s \leq x \\ 1 & \text{if } v_s > x \end{cases}$$



Claim: $\sigma(v_b)$ is optimal given β.

$$
\begin{aligned}
Eu_s(s, \beta|v_s) &= Prob(\beta(v_b) \geq s) \cdot (\lambda \cdot E[\beta(v_b)|b(v_b) \geq s] + (1-\lambda) \cdot s) \\
&+ Prob(\beta(v_b) < s) \cdot v_s
\end{aligned}
$$

$$
Prob(\beta(v_b) \geq s) = \begin{cases} 1 & \text{if } s = 0 \\ 1 - x & \text{if } 0 < s \leq x \\ 0 & \text{if } s > x \end{cases}
$$

$$
E[\beta(v_b)|\beta(v_b) \geq s] = \begin{cases} 0 \cdot x + x \cdot (1-x) & , \text{if } s = 0 \\ x & , \text{if } 0 < s \leq x \\ \emptyset & , \text{if } s > x \end{cases}
$$

$$
Eu_s(s, \beta(v_b)|v_s) = \begin{cases} 1 \cdot [\lambda \cdot x \cdot (1-x) + (1-\lambda) \cdot 0] + 0 \cdot v_s & , \text{if } s = 0 \\ (1-x) \cdot [\lambda \cdot x + (1-\lambda) \cdot s] + x \cdot v_s & , \text{if } 0 < s \leq x \\ 0 \cdot [\lambda \cdot \emptyset + (1-\lambda) \cdot s] + 1 \cdot v_s & , \text{if } s > x \end{cases}
$$

If $0 < s \leq x$, then $s = x$ is optimal.
If $x < s$, then any s is optimal.

$$\Rightarrow Eu_s(s, \beta(v_b)|v_s) = \begin{cases} \lambda \cdot x \cdot (1-x) & \text{, if } s = 0 \\ (1-x) \cdot x + x \cdot v_s & \text{, if } s = x \\ v_s & \text{, if } s > x \end{cases}$$

$\rightarrow s = 0$ is worse than $s = x$!
(One) best reply: $\sigma(v_s) = x$, if $v_s \leq x$ and $\sigma(v_s) = 1$, if $v_s > x$.

Claim: $\beta(v_b)$ is optimal given σ.

$$\begin{aligned} Eu_b(b, \sigma(v_s)|v_b) &= Prob(\sigma(v_s) \leq b) \cdot (v_b - \lambda \cdot b - (1-\lambda) \cdot E[\sigma(v_s)|\sigma(v_s) \leq b \\ &+ Prob(\sigma(v_s) > b) \cdot 0 \end{aligned}$$

$$Prob(\sigma(v_s) \leq b) = \begin{cases} 0 & \text{if } b < x \\ x & \text{if } x \leq b < 1 \\ 1 & \text{if } b \geq 1 \end{cases}$$

$$E[\sigma(v_s)|\sigma(v_s) \leq b] = \begin{cases} \emptyset & \text{if } b < x \\ x & \text{if } x \leq b < 1 \\ x^2 + 1 - x & \text{if } b \geq 1 \end{cases}$$

$$Eu_b(b, \sigma(v_s)|v_b) = \begin{cases} 0 \cdot (v_b - \lambda \cdot b - (1-\lambda) \cdot \emptyset) & \text{if } b < x \\ x \cdot (v_b - \lambda \cdot b - (1-\lambda) \cdot x) & \text{if } x \leq b < 1 \\ 1 \cdot \big(v_b - \lambda \cdot b - (1-\lambda) \cdot (x^2 + 1 - x)\big) & \text{if } b \geq 1 \end{cases}$$

If $b < x$, then any b is optimal.
If $x \leq b < 1$ then $b = x$ optimal.
If $b \geq 1$ then $b = 1$ optimal.

$$\Rightarrow Eu_b(b, \sigma(v_s)|v_b) = \begin{cases} 0 & \text{if } b < x \\ x \cdot (v_b - x) & \text{if } b = x \\ v_b - \lambda - (1-\lambda) \cdot (x^2 + 1 - x) & \text{if } b = 1 \end{cases}$$

$b = x \succsim_b b < x \;\Leftrightarrow$

$$x \cdot (v_b - x) \geq 0 \Leftrightarrow v_b \geq x$$

$b = x \succ_b b = 1 \;\Leftrightarrow$

$$v_b \cdot x - x^2 > v_b - \lambda - (1-\lambda) \cdot (x^2 + 1 - x) \Leftrightarrow v_b < 1 + \lambda \cdot x$$

$\Rightarrow b \geq 1$ is worse than $b = x$!
Hence the buyer prefers $\beta(v_b) = x$ to any other bid, if $v_b \geq x$ and $\beta(v_b) = 0$ is one best reply (of many), if $v_b < x$. □

Remarks

The equilibrium strategies do not depend on $\lambda \in (0, 1)$.
For each $x \in (0, 1)$, there is a one-price equilibrium!
Players need to coordinate on a value for x ex ante.

Which x is preferred by the seller?
Two cases: ex ante & interim.

Ex ante (seller does not know v_s):

$$\begin{aligned} E\pi_s(s(v_s), b(v_b)) &= prob\{v_s \leq x\} \cdot ((1-x) \cdot x + x \cdot E[v_s | v_s \leq x]) \\ &+ prob\{v_s > x\} \cdot E[v_s | v_s > x] \\ &= \frac{1}{2} \cdot \left(1 - x^3 + x^2\right) \qquad \Rightarrow x_s^* = \frac{2}{3} \end{aligned}$$

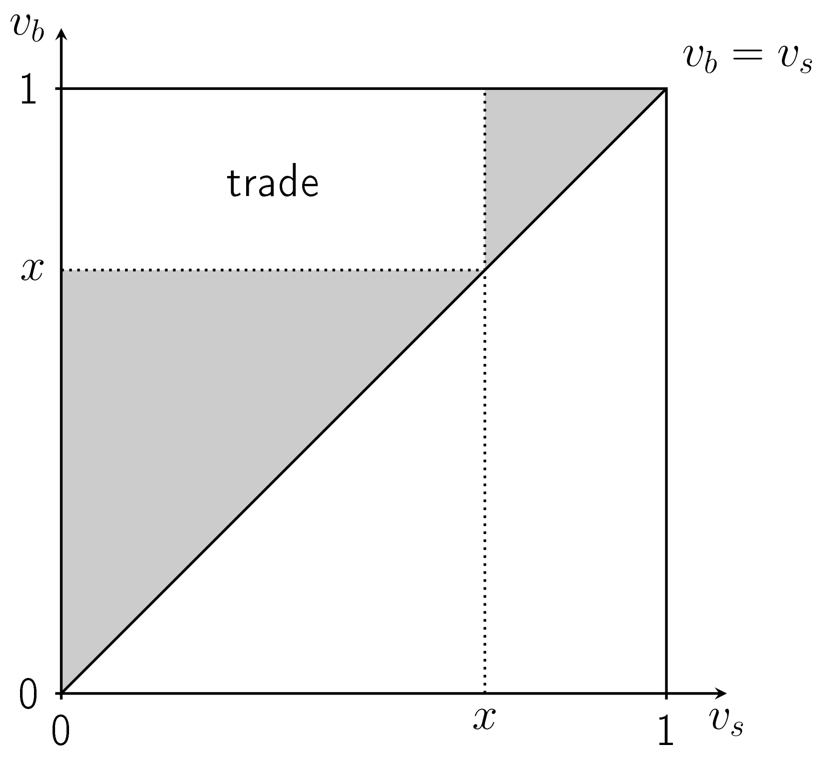
Interim (seller knows v_s):

$$E\pi_s(s(v_s), b(v_b)|v_s) = \begin{cases} (1-x) \cdot x + x \cdot v_s & \text{if } v_s \leq x \;\rightarrow\; x_s(v_s) = \frac{v_s+1}{2} \\ v_s & \text{if } v_s > x \;\rightarrow x_s(v_s) = 0 \end{cases}$$

$$x_s(v_s) = \frac{v_s+1}{2} \succ_s x_s(v_s) = 0 \Leftrightarrow \left(\frac{v_s+1}{2}\right)^2 > v_s \checkmark$$

$$\Rightarrow x_s^*(v_s) = \frac{v_s+1}{2}$$

Is the equilibrium efficient? no!



Shaded areas: trade would be efficient but does not take place.

gains from trade – one price equilibria

For each trade the gain of trade is $v_b - v_s$. The expected gain from trade is

$$\int_x^1 \int_0^x (v_b - v_s) dv_s dv_b = \frac{1}{2} \cdot (1-x) \cdot x$$

which is maximal at $x = \frac{1}{2}$ for which the expected gain equals $\frac{1}{8}$.

Exercise

Check whether there are one price equilibria with $x = 0$ or $x = 1$!

Affine Equilibria

Suppose the buyer uses the strategy

$$\beta(v_b) = \gamma_b + \delta_b \cdot v_b \text{ with } \gamma_b \geq 0, \delta_b > 0 \ .$$

Then the buyers's bid β is uniformly distributed on $[\gamma_b, \gamma_b + \delta_b]$ and

$$Prob(s \leq \beta(v_b)) = \begin{cases} 1 & \text{if } s \leq \gamma_b \\ \frac{\gamma_b + \delta_b - s}{\delta_b} & \text{if } \gamma_b < s < \gamma_b + \delta_b \\ 0 & \text{if } s \geq \gamma_b + \delta_b \end{cases}$$

$$E[\beta(v_b) | s \leq \beta(v_b)] = \begin{cases} \gamma_b + \delta_b \cdot \frac{1}{2} & \text{if } s \leq \gamma_b \\ \frac{s + \gamma_b + \delta_b}{2} & \text{if } \gamma_b < s < \gamma_b + \delta_b \\ \emptyset & \text{if } s \geq \gamma_b + \delta_b \end{cases} \ .$$

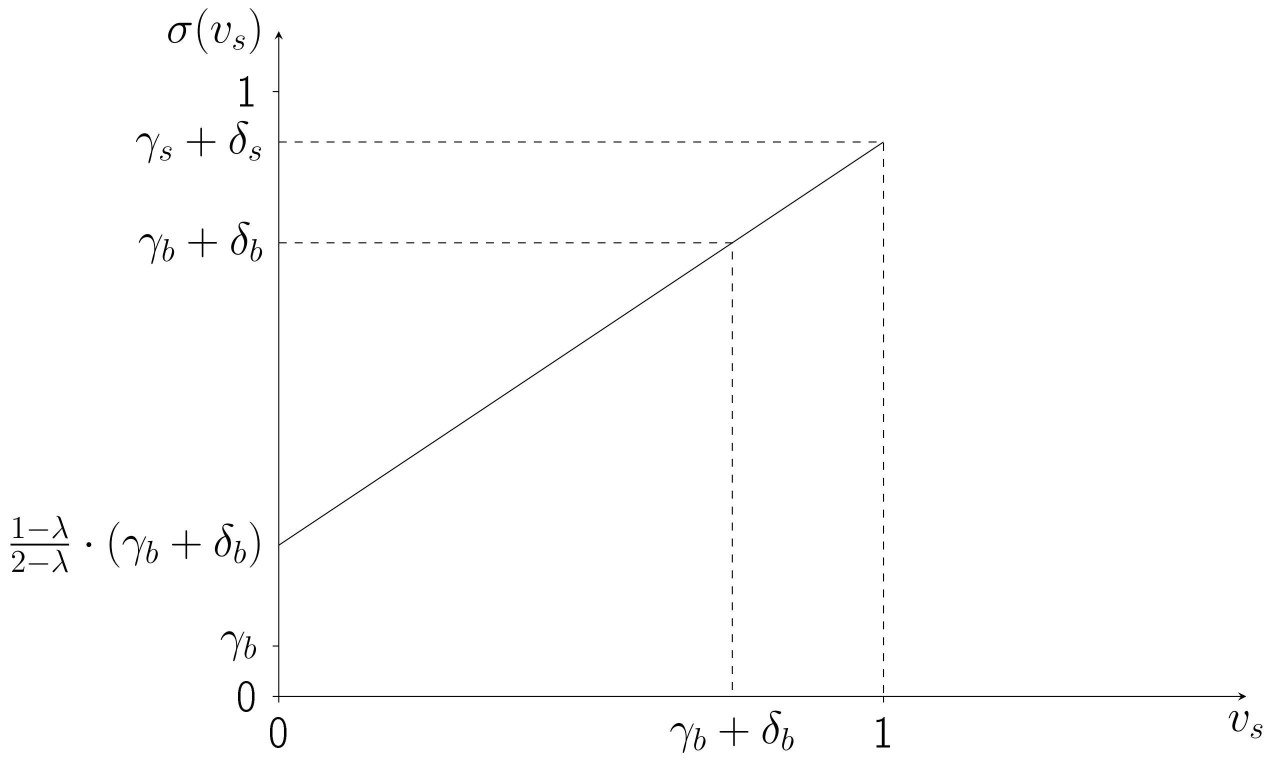
The seller expects

$$Eu_s(s,\beta(v_b)|v_s) = \begin{cases} 1\cdot\left[\lambda\cdot(\gamma_b+\frac{1}{2}\cdot\delta_b)+(1-\lambda)\cdot s)\right] & \text{, if } s\le\gamma_b \\ \left(1-\frac{s-\gamma_b}{\delta_b}\right)\cdot\left[\lambda\cdot\frac{s+\gamma_b+\delta_b}{2}+(1-\lambda)\cdot s\right] & \\ +\frac{s-\gamma_b}{\delta_b}\cdot[v_s] & \text{, if } \gamma_b<s<\gamma_b+\delta_b \\ 0\cdot[\lambda\cdot ?+(1-\lambda)\cdot s]+1\cdot[v_s] & \text{, if } s\ge\gamma_b+\delta_b \end{cases}$$

If $s \geq \gamma_b + \delta_b$, then any s is optimal.
If $s \leq \gamma_b$ then $s = \gamma_b$ is optimal.

If $\gamma_b < s < \gamma_b + \delta_b$:

The first order condition implies $\sigma(v_s) = \frac{1-\lambda}{2-\lambda} \cdot (\delta_b + \gamma_b) + \frac{1}{2-\lambda} \cdot v_s$.



Note that $s(v_s = \gamma_b + \delta_b) = \gamma_b + \delta_b$ and that $\sigma(v_s) > \gamma_b + \delta_b \ \forall \ v_s > \gamma_b + \delta_b$. Further if $v_s \geq \gamma_b - (1-\lambda)\cdot\delta_b$, then $\sigma(v_s) \geq \gamma_b$. Assume $\gamma_b - (1-\lambda)\cdot\delta_b < 0$.

Then

$$\sigma(v_s) = \frac{1-\lambda}{2-\lambda} \cdot (\gamma_b + \delta_b) + \frac{1}{2-\lambda} \cdot v_s \; \forall \; v_s \in [0,1]$$

Suppose the seller's strategy is

$$\sigma(v_s) = \gamma_s + \delta_s \cdot v_s \text{ with } \gamma_s \geq 0, \; \delta_s > 0 \; .$$

Then the seller's ask is uniformly distributed on $[\gamma_s, \gamma_s + \delta_s]$ and The probability that a trade takes place is

$$\begin{aligned} Prob(\sigma(v_s) \leq b) &= Prob(\gamma_s + \delta_s \cdot v_s \leq b) \\ &= \begin{cases} 0 & \text{if } b \leq \gamma_s \\ \frac{b-\gamma_s}{\delta_s} & \text{if } \gamma_s < b < \gamma_s + \delta_s \\ 1 & \text{if } b \geq \gamma_s + \delta_s \end{cases} . \end{aligned}$$

$$E[\sigma(v_s)|b \geq \sigma(v_s)] = \begin{cases} \emptyset & \text{if } b \leq \gamma_s \\ \frac{\gamma_s + b}{2} & \text{if } \gamma_s < b < \gamma_s + \delta_s \\ \gamma_s + \delta_s \cdot \frac{1}{2} & \text{if } b \geq \gamma_s + \delta_s \end{cases}$$

The bidder's payoff:

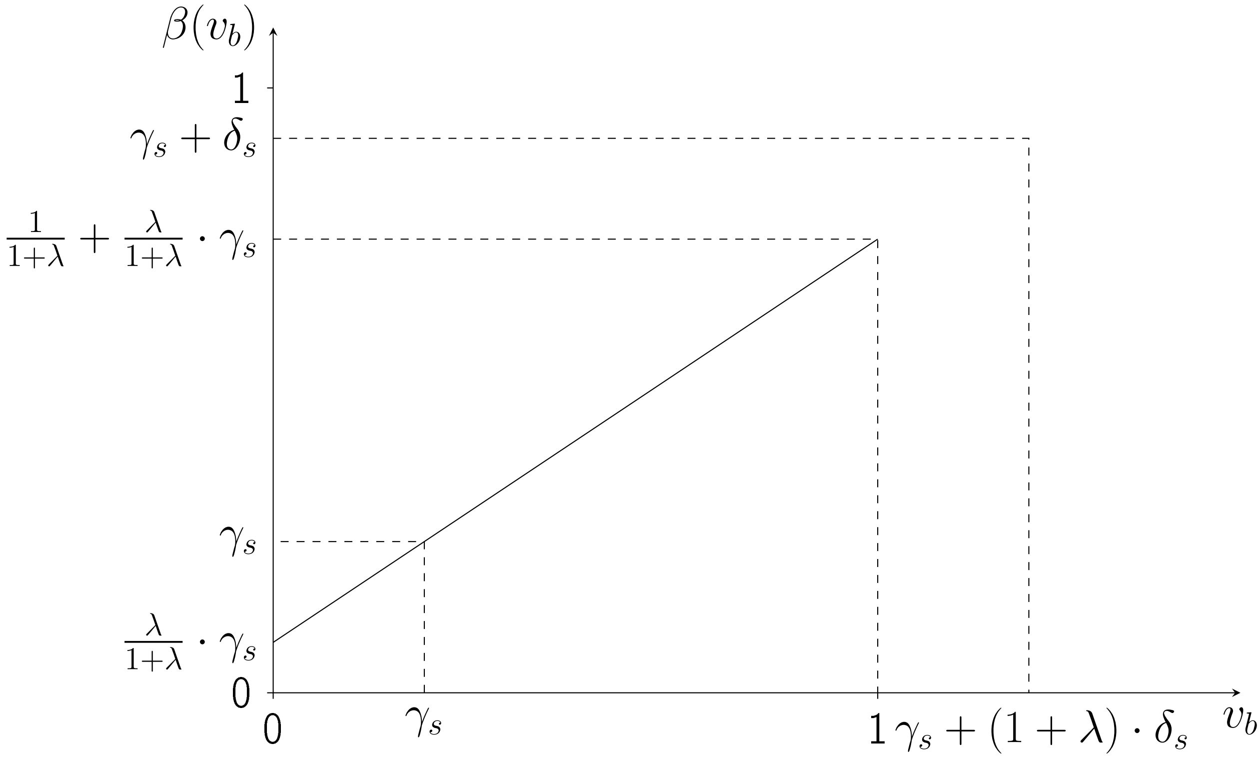
$$\begin{aligned} & E\pi_b(b, \sigma(v_s)|v_b) \\ = & \begin{cases} 0 & \text{, if } b \leq \gamma_s \\ \frac{b-\gamma_s}{\delta_s} \cdot \left[v_b - \lambda \cdot b - (1-\lambda) \cdot \frac{\gamma_s + b}{2} \right] & \text{, if } \gamma_s < b < \gamma_s + \delta_s \\ 1 \cdot \left[v_b - \lambda \cdot b - (1-\lambda) \cdot (\gamma_s + \frac{1}{2}\delta_s) \right] & \text{, if } b \geq \gamma_s + \delta_s \end{cases} \end{aligned}$$

$b \leq \gamma_s$: any such b is optimal.
$b \geq \gamma_s + \delta_s$: $b = \gamma_s + \delta_s$ is optimal.
$\gamma_s < b < \gamma_s + \delta_s$:

The first order condition yields $\beta(v_b) = \frac{\lambda}{\lambda+1} \cdot \gamma_s + \frac{1}{\lambda+1} \cdot v_b$.

The second order condition is satisfied ($-\frac{1+\lambda}{\delta_s} < 0$).

$$Eu_b\left(\beta(v_b) = \frac{\lambda}{1+\lambda}\cdot\gamma_s + \frac{1}{1+\lambda}\cdot v_b, \sigma(v_s)\middle| v_b\right) = \frac{(v_b-\gamma_s)^2}{2\cdot(1+\lambda)\cdot\delta_s} \geq 0$$



Note that $b(v_b = \gamma_s) = \gamma_s$. Hence for $v_b \leq \gamma_s$ it is optimal to bid $\beta(v_b) \leq \gamma_s$ at which the probability of trade is zero. Further, if $v_b = \gamma_s + (1+\lambda)\cdot\delta_s$, then $\beta(v_b) = \gamma_s + \delta_s$. Hence $\beta(v_b) = \gamma_s + \delta_s$ is optimal for all $v_b > \gamma_s + (1+\lambda)\cdot\delta_s$. Assume $\gamma_s + (1+\lambda)\cdot\delta_s > 1$.

$$\Rightarrow \beta(v_b) = \frac{\lambda}{1+\lambda}\cdot\gamma_s + \frac{1}{1+\lambda}\cdot v_b \;\forall\; v_b \in [0,1]$$

The bidding strategies

$$\beta(v_b) = \frac{\lambda}{\lambda+1} \cdot \gamma_s + \frac{1}{\lambda+1} \cdot v_b$$
$$\sigma(v_s) = \frac{1-\lambda}{2-\lambda} \cdot (\delta_b + \gamma_b) + \frac{1}{2-\lambda} \cdot v_s$$

imply that

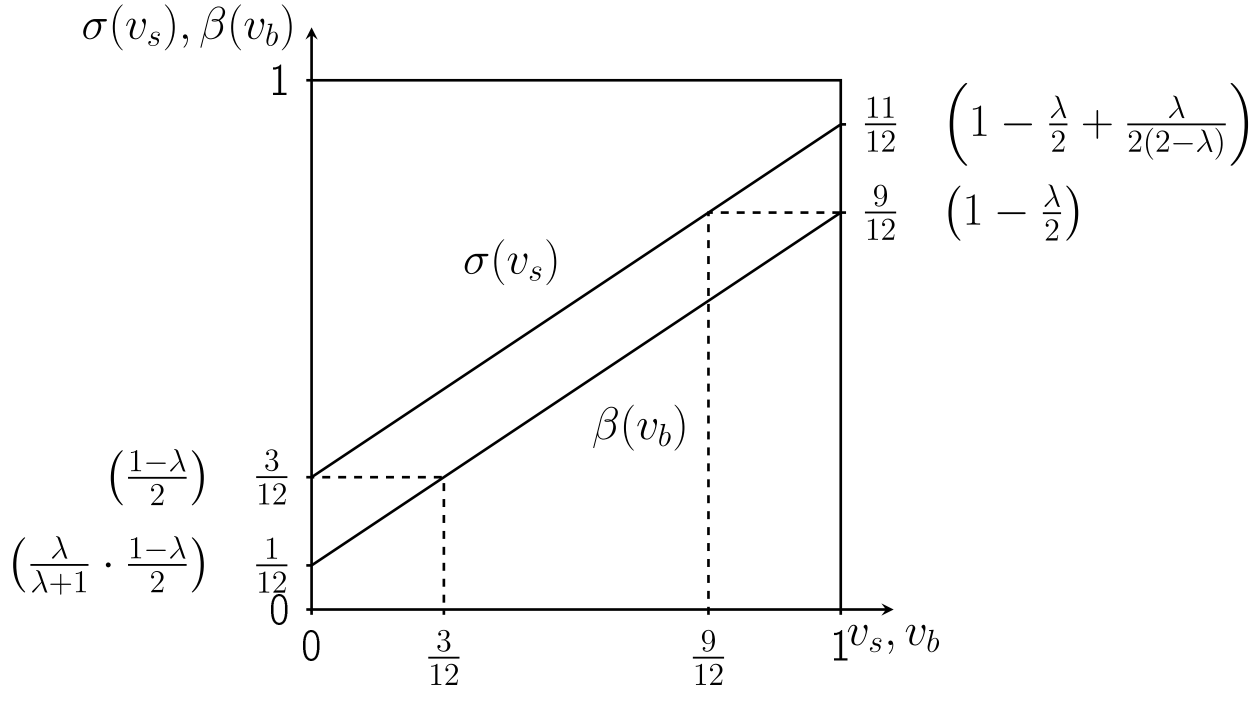
$$\gamma_b = \frac{\lambda}{\lambda+1} \cdot \frac{1-\lambda}{2} \qquad \delta_b = \frac{1}{\lambda+1}$$

$$\gamma_s = \frac{1-\lambda}{2} \qquad \delta_s = \frac{1}{2-\lambda}$$

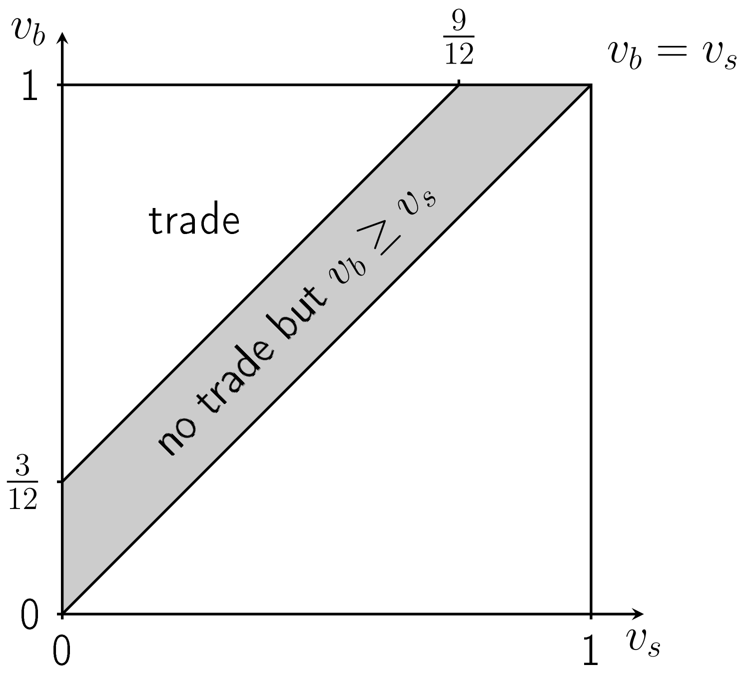
Remarks

- $\gamma_b < \gamma_s$ and $\gamma_b + \delta_b < \gamma_s + \delta_s \ \forall \ \lambda \in (0,1)$
- The efficient strategies $\beta(v_b) = v_b$, $\sigma(v_s) = v_s$ are not individually rational.

Illustration for $\lambda = \frac{1}{2}$: $\beta(v_b) = \frac{1}{12} + \frac{8}{12} \cdot v_b \qquad \sigma(v_s) = \frac{3}{12} + \frac{8}{12} \cdot v_s$



Trade does not take place if $v_b < \frac{3}{12}$ or if $v_s > \frac{9}{12}$:



expected gain from trade – affine equilibrium

$$\frac{1}{6} > \int_{\frac{1}{4}}^{1} \int_{0}^{v_b - \frac{1}{4}} v_b - v_s dv_s dv_b = \frac{9}{64} > \frac{1}{8}$$

· Wilson (1985): multi-buyer/multi-seller generalization. If there are sufficiently many buyers and sellers, the auction is efficient.

Chapter 3

The '3G' Mobile-Phone Auctions

Taken from Klemperer 'Auctions: Theory and Practice' chapter 5
Resp. Klemperer (2002) 'How (not) to run auctions: The European 3G telecom auctions' *EER* and Klemperer (2002) 'What really matters in auction design' *JEP*

3.1 Overview of the European Auctions

The good that was sold was quite similar: blocks of spectrum

Ex ante expectations:

- roughly constant per capita value
- smaller countries (-)
- centrally located countries (+)
- richer countries (+)

	2000		2001
United Kingdom (March / April) (ascending)	650	Belgium (March)	45
Netherlands (July) (ascending)	170	Greece (July)	45
Germany (July / August)	615	Denmark (September)	95
Italy (October) (ascending)	240		
Austria (November)	100		
Switzerland (November / December) (ascending)	20		

Revenue in per capita Euros of the European '3rd generation' mobile telecommunication (or UMTS) licences

Sequence of auctions: the market sentiment cooled over time.

What does really matter in aution design?

· attract entry!
· prevent collusion!

$\rightarrow$ in industrial organization, the social planner has the same goals!

- disadvantages of *ascending auctions*:

  bidders use early rounds to signal

  bidders use later rounds to punish those who fail to cooperate

  entry deterrence: weaker potential bidders know that stronger bidder can always top bids

- advantages of *sealed bid auctions*:

  neither signalling nor punishment possible

  topping an opponent's bid is more costly

- disadvantages of *sealed bid auctions*:

  more likely that a low valuation bidder wins → inefficient outcome

The UK Auction (March-April 2000)

- world's first 3G auction
- plan: sell four licences
- problem: four (2G-)incumbents

  with customer bases, lower costs of building infrastructure

  → entry deterrence!

- hybrid auction design:

  ascending auction until only five bidders remain

  after which the survivors submit sealed bids (above current price)

  the four highest bids receive a licence at the fourth highest bid.

- design performed extremely well in experiments
- during the planning period, five licences became available
- → ascending auction

  each bidder could only win one licence

licences were indivisible

→ collusion difficult

- one licence for a non-incumbent: carrot to attract entrants
- result: nine new entrants, revenue of 39 billion euros (650 p. c.)

See Binmore and Klemperer (2002)

The Netherlands Auction (July 2000)

- four strong and one weak incumbent
- five licences
- ascending auction
- dysfunctional competition policy

  → weak incumbent joined with Deutsche Telekom

  → very few entrants would show up in the auction
- in the end one entrant (Versatel) showed up but stopped bidding after receiving a legal threat by one of the incumbents (Telfort).
- Government failure:

  no action against Telfort

  no minimum prices
- result: no new entrant, revenue of 3 billion euros (170 p. c.)
- sealed bid auction would have been better!

The German Auction (July-August 2000)

- twelve blocks of spectrum, bidders could create licences of either two or three blocks.

- at most one licence for each firm

- possible outcomes:

  four winners with each three blocks

  five winners: two with three blocks, three with two blocks

  six winners with each two blocks

- ascending auction

- the number of winners

  depends on bidders, who might have better info than the government

  does not depend on consumers' interests

- complexity of this auction could have caused problems, but the government was lucky.

- two incumbents

  Deutsche Telekom, 40% Vodafone-Mannesmann, 40%

- seven bidders participated

- collusive offer by MobilCom:

'Should [Debitel] fail to secure a license [it could] become a 'virtual network operator' using MobilCom's network while saving on the cost of the license'.

- No punishment by government
- Again: no reserve price.
- Debitel quitted not immediately, but at a relatively low level
- six bidders left → two strategies for incumbents

  raise price to force two of the weaker firms to quit

  → high auction revenues but concentrated industry

  tacit collusion

  → low auction revenue but competitive industry
- Vodafone-Mannesmann endet a number of bids with the digit '6'
- Deutsche Telekom pushed up the price but giving up at the UK-reserve price level of the weaker competitors.

  Telekom is owned by the government!
- result: high revenues & unconcentrated industry!

Jehiel and Moldovanu (2001)
Grimm et al (2001)

The Italian Auction (October 2000)

- Same design as UK-auction
- additional rule

  if there are not enough bidders in the prequalification, then the number of licences can be reduced.
- firms had learned from the earlier auctions who were the strongest bidders

  → much less entrants as compared with the UK auction
- six bidders entered to compete for five licences
- one bidder quit after less than two days
- result: 14 billion euros
- a sealed bid auction would have been better!

The Swiss Auction (November / December 2000)

- Same design as UK-auction
- four licences
- four bidders
- very low reserve price
- result: desaster. the four bidders had to pay the reserve price.

Chapter 4

General Symmetric Private Values

Taken from Vijay Krishna 'Auctions Theory' (2009), chapter two.

- single object for sale
- $N \in \mathbb{N}$ potential bidders, $N \geq 2$
- bidder $i = 1, \ldots, N$ has valuation V_i
- each V_i is iid on $[0, \omega]$ according to cdf F (with $\omega \in \mathbb{R}_+$)
- F has continuous density $f = F'$ and has full support.
- $E[V_i] < \infty$
- Bidder i knows
  realization v_i of V_i
  other bidders' values are iid according to F
- no liquidity constraints, risk neutral bidders
- All this is common knowledge.
- strategy for bidder i: $\beta_i : [0, \omega] \to \mathbb{R}_+$

Subject of Analysis & Research Question

We analyze

1. A first-price sealed-bid auction, where the highest bidder gets the object and pays the amount he bids.

2. A second-price sealed-bid auction, where the highest bidder gets the object and pays the second highest bid.

We ask

- What are symmetric equilibrium strategies in a first-price auction (1.) and a second-price auction (2.)?

- From the point of view of the seller, which of the two auction formats yields a higher expected selling price in equilibrium?

Given the bids $b = (b_1, \ldots, b_N)$ and given the bid b_i of bidder i, define $\hat{b}_{-i} := \max\limits_{j \neq i} b_j$. Given the random variables $V = (V_1, \ldots, V_N)$, define $\hat{V}_{-i} := \max\limits_{j \neq i} V_j$.

Second Price Auctions

Given the realization v_i of bidder i the payoffs are

$$u_i(b_i, b_{-i}|v_i) = \begin{cases} v_i - \hat{b}_{-i}\ b_j & \text{, if } b_i > \hat{b}_{-i} \\ \frac{1}{|\{j : b_j = b_i\}|} \cdot (v_i - b_i) & \text{, if } b_i = \hat{b}_{-i} \\ 0 & \text{, if } b_i < \hat{b}_{-i} \end{cases}$$

Proposition:

$\beta(v) = v$ is a weakly dominant strategy.

Proof:

See proof for the case $N = 2$ and $F(v) = v$. (The proof makes no use of the number of bidders and the distribution of valuations, evcept for private values.)

Expected Payments in Second Price Auctions

Given that $\beta_i(v_i) = v_i \;\forall\; i$, the probability that bidder i with bid v_i wins:

$$Prob\left(\max_{j\neq i} V_j < v_i\right) = \prod_{j\neq i} Prob\,(V_j < v_i) = \prod_{j\neq i} F(v_i) = F(v_i)^{N-1} =: G(v_i)$$

$G(v_i)$ is the cdf of the second highest valuation given that v_i is the highest valuation. Density: $g(v_i) = \frac{dG(v_i)}{dv_i}$

Expected price conditional on $v_i > V_j \;\forall\; j \neq i$:

$$E\left[\hat{V}_{-i}\middle|\, \hat{V}_{-i} < v_i\right] = \frac{1}{G(v_i)} \cdot \int_0^{v_i} y dG(y)$$

$\Rightarrow$ bidder i with valuation and bid v_i expects to pay

$$G(v_i) \cdot E\left[\hat{V}_{-i}\middle|\, \hat{V}_{-i} < v_i\right] = \int_0^{v_i} y dG(y)\ .$$

First-Price Auctions

A bidder with valuation v_i and bid b_i has payoff

$$u_i(b_i, b_{-i}|v_i) = \begin{cases} v_i - b_i & \text{, if } b_i > \hat{b}_{-i} \\ \frac{1}{|\{j:b_j=b_i\}|} \cdot (v_i - b_i) & \text{, if } b_i = \hat{b}_{-i} \\ 0 & \text{, if } b_i < \hat{b}_{-i} \end{cases}\ .$$

Symmetric Equilibrium

Suppose all bidders $j \neq i$ use the same strictly increasing and differentiable strategy $\beta : [0, \omega] \to \mathbb{R}_+$.

Bidder i with value v_i should not bid $b_i > \overset{\text{highest potential bid}}{\beta(\omega)}$! $\Rightarrow \beta_i(v_i) \leq \beta(\omega)$.

Bidder i with value 0 should not bid $b_i > 0!$ $\Rightarrow \beta_i(0) = 0$.

Bidder i wins the auction whenever $\max\limits_{j \neq i} \beta(V_j) < \beta_i(v_i)$.

$\beta(\cdot)$ strictly increasing $\Rightarrow \max\limits_{j \neq i} \beta(V_j) = \beta\left(\max\limits_{j \neq i} V_j\right) = \beta\left(\hat{V}_{-i}\right)$.

Probability of Winning

Bidder i with bid b_i wins the auction, if $\hat{V}_{-i} < \beta^{-1}(b_i)$.

$$Prob\left(\hat{V}_{-i} < \beta^{-1}(b_i)\right) = G\left(\beta^{-1}(b_i)\right)$$

Necessary Condition for Optimality

$\Rightarrow$ bidder i with valuation v_i and bid b_i expects payoffs:

$$G\left(\beta^{-1}(b_i)\right) \cdot (v_i - b_i)$$

First order condition:

$$g\left(\beta^{-1}(b_i)\right) \cdot \frac{1}{\beta'\left(\beta^{-1}(b_i)\right)} \cdot (v_i - b_i) - G\left(\beta^{-1}(b_i)\right) = 0$$

Assume $b_i = \beta(v_i)$ (symmetry) and rearrange:

$$\begin{aligned}
\Rightarrow g(v_i) \cdot v_i &= G(v_i) \cdot \beta'(v_i) + g(v_i) \cdot \beta(v_i) = \frac{d}{dv}\left(G(v_i) \cdot \beta(v_i)\right) \\
\Rightarrow \beta(v_i) &= \frac{1}{G(v_i)} \cdot \int_0^{v_i} g(y) \cdot y \, dy \\
&= E\left[\hat{V}_{-i} \middle| \hat{V}_{-i} < v_i\right]
\end{aligned}$$

Sufficiency

Suppose bidder i with valuation v_i deviates and bids b_i instead of $\beta(v_i)$. Define $z_i = \beta^{-1}(b_i)$.

If the other players believe that bidder i uses $\beta(\cdot)$, then they believe that the realization of V_i is z_i.

$$\begin{aligned}
EU_i(b_i, \beta | v_i) &= G(z_i) \cdot (v_i - \beta(z_i)) \\
&= G(z_i) \cdot v_i - \int_0^{z_i} y \cdot g(y) dy \\
&= G(z_i) \cdot v_i - \left(G(z_i) \cdot z_i - \int_0^{z_i} G(y) dy \right) \\
&= G(z_i) \cdot (v_i - z_i) + \int_0^{z_i} G(y) dy
\end{aligned}$$

If bidder i does not deviate ($b_i = \beta(v_i) \Rightarrow z_i = \beta^{-1}(\beta(v_i)) = v_i$), he expects

$$EU_i(\beta(v_i), \beta | v_i) = G(v_i) \cdot (v_i - \beta(v_i)) = \int_0^{v_i} G(y) dy$$

Hence

$$\begin{aligned} EU_i(\beta(v_i), \beta|v_i) - EU_i(\beta(z_i), \beta|v_i) &= \int_0^{v_i} G(y)dy \\ &\quad -G(z_i) \cdot (v_i - z_i) - \int_0^{z_i} G(y)dy \\ &= G(z_i) \cdot (z_i - v_i) + \int_{z_i}^{v_i} G(y)dy \\ &\geq G(z_i) \cdot (z_i - v_i) + \int_{z_i}^{v_i} G(z_i)dy \\ &= G(z_i) \cdot (z_i - v_i) + G(z_i) \cdot (v_i - z_i) = 0 \end{aligned}$$

Grafical 'proof':

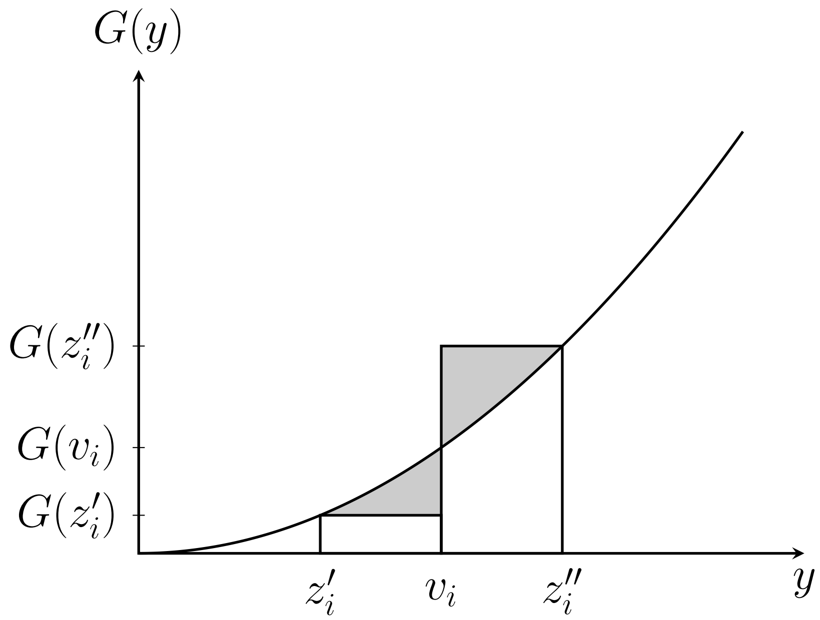


Figure 4.1: The gray areas depict the losses from over- and underbidding.

$$EU_i(\beta(v_i), \beta|v_i) > EU_i(b_i, \beta|v_i) \;\forall\; b_i \neq \beta(v_i)$$

Lemma

$$\beta(v_i) < v_i \;\forall v_i \in (0, \omega]$$

Proof

Clearly, $y \cdot g(y) < G(y) + y \cdot g(y) \; \forall \, y \in (0, \omega]$. As $G(y) + y \cdot g(y) = (y \cdot G(y))'$ we have that

$$\int_0^{v_i} y \cdot g(y) dy < v_i \cdot G(v_i) \Leftrightarrow \frac{1}{G(v_i)} \cdot \int_0^{v_i} y \cdot g(y) dy < v_i$$

□

Exercise: calculate the equilibrium bid function for N bidders with uniformly distributed valuations!

Expected Payments in First-Price Auctions

Bidder i with valuation v_i and bid $\beta(v_i)$ expects to pay

$$G(v_i) \cdot \beta(v_i) = G(v_i) \cdot E\left[\hat{V}_{-i} \middle| \hat{V}_{-i} < v_i\right] = \int_0^{v_i} y dG(y)$$

Proposition

With independently and identically distributed private values, the expected revenue in a first-price auction is the same as the expected revenue in a second-price auction.

Reserve Prices in Second-Price Auctions

Consider reserve price $r > 0$.

A bidder with $v_i \leq r$ cannot gain from winning
$\Rightarrow \beta_i(v_i) = v_i$ is optimal.

For $v_i > r$: $\beta_i(v_i) = v_i$ is weakly dominant.

Expected payment of bidder i with valuation $v_i \geq r$:

$$G(v_i) \cdot \max\left\{E\left[\hat{V}_{-i}\middle|\hat{V}_{-i} < v_i\right], r\right\} = r \cdot G(r) + \int_r^{v_i} y \cdot g(y)\ dy$$

Reserve Prices in First-Price Auctions

Consider reserve price $r > 0$.

A bidder with $v_i \leq r$ cannot gain from winning.

Symmetric equilibrium strategy for $v_i \geq r$: (without derivation)

$$\beta(v_i) = \max\left\{E\left[\hat{V}_{-i}\middle|\hat{V}_{-i} < v_i\right], r\right\}$$

$\Rightarrow$ Expected payment of bidder with $v_i \geq r$:

$$G(v_i) \cdot \beta(v_i) = r \cdot G(r) + \int_r^{v_i} y \cdot g(y)\ dy$$

$\Rightarrow$ Revenue equivalence does only hold if reserve prices are the same!
No general revenue equivalence!

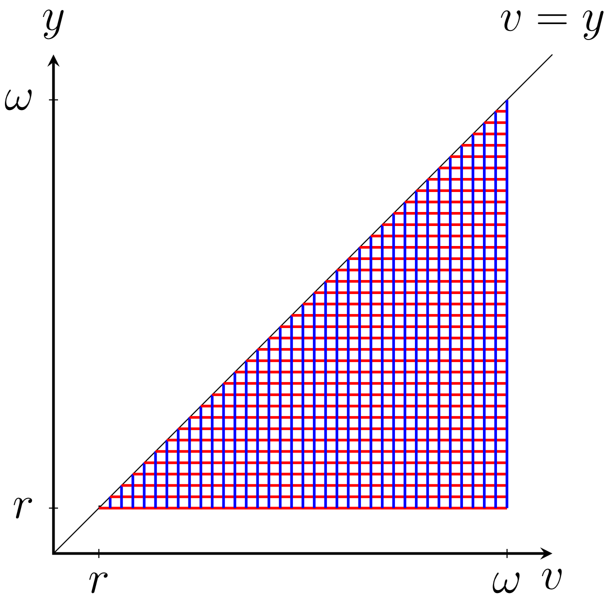
Revenue Effects of Reserve Prices

Ex ante expected payment of bidder i:

$$\begin{aligned}
& \int_r^{\omega}\left(r \cdot G(r) + \int_r^{v_i} y \cdot g(y)\ dy\right) f(v_i) dv_i \\
= \ & r \cdot G(r) \cdot (1 - F(r)) + \int_r^{\omega}\int_r^{v_i} y \cdot g(y)\ dy\ f(v_i)\ dv_i \\
= \ & r \cdot G(r) \cdot (1 - F(r)) + \int_r^{\omega} y \cdot g(y) \cdot (1 - F(y))\ dy
\end{aligned}$$

Interchanging the Order of Integration

$$\int_r^\omega \int_r^{v_i} y \cdot g(y) \cdot f(v_i)\, dy \cdot dv_i = \int_r^\omega \int_y^\omega y \cdot g(y) \cdot f(v_i)\, dv_i \cdot dy$$



Expected revenue:

$$N \cdot \left(r \cdot G(r) \cdot (1 - F(r)) + \int_r^\omega y \cdot g(y) \cdot (1 - F(y))\, dy \right)$$

Optimal Reserve Price

Suppose the seller attaches a value $v_0 \in [0, \omega)$ to the object.

Clearly: $r \geq v_0$!

$\Rightarrow$ the seller maximizes

$$N \cdot \left(r \cdot G(r) \cdot (1 - F(r)) + \int_r^\omega y \cdot g(y) \cdot (1 - F(y))\, dy \right) + F(r)^N \cdot v_0 \;.$$

First order condition:

$$N \cdot (G(r) \cdot (1 - F(r)) - G(r) \cdot r \cdot f(r)) + N \cdot \underbrace{F(r)^{N-1}}_{G(r)} \cdot f(r) \cdot v_0 = 0$$

$$\Rightarrow \frac{f(r^*)}{1-F(r^*)} \cdot (r^* - v_0) = 1$$

Hazard Rate $\lambda(v) = \frac{f(v)}{1-F(v)}$

Example

$F(\cdot)$: life distribution
$\to \lambda(t)dt$ conditional probability of death in $[t, t+dt]$ given survival until t.

Remarks:

$F(\cdot)$ has full support
$\Rightarrow \lambda(v) > 0 \;\forall\; v$

Solutions to the First Order Condition

- $v_0 = 0 \Rightarrow$

$$\left.\frac{\partial^2 EU_0}{(\partial r)^2}\right|_{r=v_0} = N \cdot g(0) > 0 \Rightarrow r^* > 0$$

- $v_0 > 0 \Rightarrow$

$$\left.\frac{\partial EU_0}{\partial r}\right|_{r=v_0} = N \cdot G(v_0) \cdot (1 - F(v_0)) > 0 \Rightarrow r^* > v_0$$

$$\Rightarrow (r^* - v_0) \cdot \lambda(r^*) = 1$$

Sufficiency?

$$\left.\frac{\partial^2 EU_0}{(\partial r)^2}\right|_{r=r^*} = N \cdot G(r^*) \cdot (-\underbrace{\lambda(r^*)}_{>0} - \underbrace{(r^* - v_0)}_{>0} \cdot \lambda'(r^*))$$

If the hazard rate is monotonic, then r^* is the optimal reserve price.

Remarks on the Optimal Reserve Price

The optimal reserve price is defined by $(r^* - v_0) \cdot \lambda(r^*) = 1$

- if $\lambda'(r^*) > 0$
- does not depend on the number of bidders!
- excludes bidders with positive probability, even if $v_0 = 0$!
- induces inefficient outcome with positive probability ($r^* > v_0$)

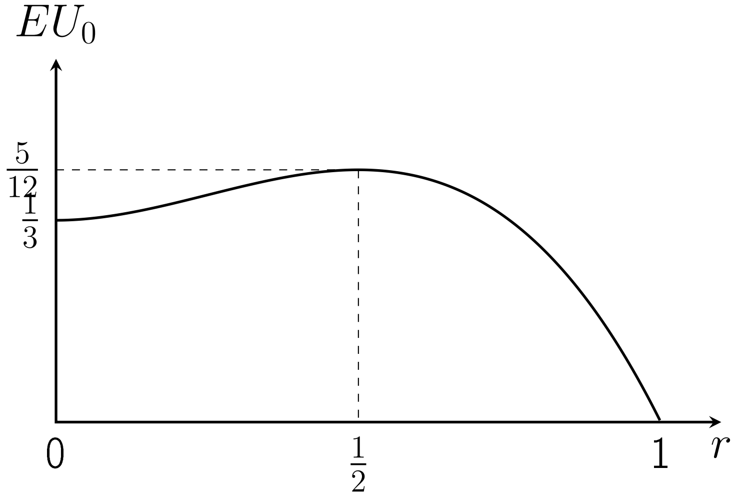
$$F(r^*)^N - F(v_0)^N$$

Optimal Reserve Price for $F(v) = v$

- $F(r) = r$, $f(r) = 1$, $\Rightarrow \lambda(r) = \frac{1}{1-r}$
- $(r^* - v_0) \cdot \frac{1}{1-r^*} = 1 \Leftrightarrow r^* = \frac{1+v_0}{2}$

The Seller's Expected Payoff and the Reserve Price for $v_0 = 0$ and $N = 2$

$$\Rightarrow EU_0(r^*) = \frac{1}{N+1} \cdot \left(N - 1 + \left(\tfrac{1}{2}\right)^N\right) = \Big|_{N=2} \frac{5}{12} > \frac{1}{3}$$



Literature

- Vickrey (1961)
  Equilibrium strategies for $F(v) = v$ in 1st & 2nd-price auctions, revenue equivalence

- Vickrey (1962)
  as above for general $F(v)$

- Myerson (1981), Riley & Samuelson (1982)
  reserve prices

Chapter 5

The Revenue Equivalence Principle

Taken from Vijay Krishna 'Auction Theory' (2009), chapters three and four.

5.1 Independent and Symmetric Values, Risk-Neutral Bidders

Definition

An auction A is a *standard auction*, if the highest bid wins the object.

Theorem

Suppose that values are independently and identically distributed, that the density is atomless, that all bidders are risk neutral and that A is a standard auction.

Then any symmetric and increasing equilibrium in which the expected payment of a bidder with value zero is zero yields the same expected revenue to the seller.

Proof

Consider an auction A and fix a symmetric and increasing equilibrium β of A.

$m^A(v)$: expected equilibrium payment by a bidder with value v.

Suppose bidder i bids $\beta(z)$ instead of $\beta(v_i)$, $z \neq v_i$.

He wins, if $z > \max_{j \neq i} V_j$.

His expected payoff is

$$EU_i^A(z, \beta | v_i) = G(z) \cdot v_i - m^A(z) \ .$$

First order condition:

$$\frac{\partial}{\partial z} EU_i^A(z, \beta | v_i) = g(z) \cdot v_i - \frac{d}{dz} m^A(z) = 0$$

In equilibrium it is optimal to report $z = v_i$, hence

$$\frac{d}{dv} m^A(v) = g(v) \cdot v \ \forall \ v \in [0, \omega]$$

and therefore

$$m^A(v) - m^A(0) = \int_0^v g(y) \cdot y \ dy = G(v) \cdot E\left[\max_{j \neq i} V_j \middle| \max_{j \neq i} V_j < v_i \right] \ \square$$

Application: Equilibrium of All-Pay Auctions

$$\begin{aligned} \beta^{AP}(v) &= m^{AP}(v) \\ &= \int_0^v y \cdot g(y) \ dy \end{aligned}$$

Expected payoff given bid $\beta(z)$ when value is v:

$$G(z) \cdot v - \int_0^z y \cdot g(y)\ dy = G(z) \cdot (v - z) + \int_0^z G(y) dy$$

which is maximal for $z = v$.

5.2 Risk-Averse Bidders

Suppose that each bidder has a utility function

$$u : \mathbb{R}_+ \to \mathbb{R}$$

with

- $u(0) = 0$
- $u' > 0$
- $u'' < 0$

and is an expected utility maximizer.

Lemma If $u(\cdot)$ is strictly concave, we have $\frac{u(y)}{u'(y)} > y\ \forall\ y > 0$.

Proof

Write $u(y) - u(0) = \int_0^y u'(s) ds$ and $u'(s) = \int_0^s u''(x) dx + u'(0)$. As $u(0) = 0$ we have $u(y) = \int_0^y \int_0^s u''(x) dx + u'(0) ds$.

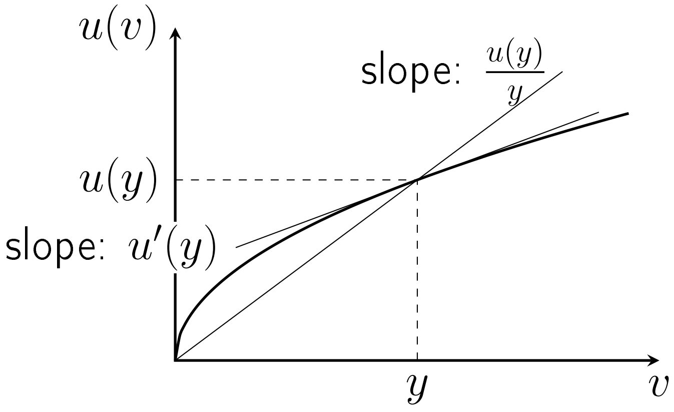
By changing the order of integration, we have

$$u(y) = \int_0^y \int_x^y u''(x) ds dx + \int_0^y u'(0) ds = \int_0^y u''(x)(y - x) dx + u'(0) \cdot y$$

Hence

$$u(y) - y \cdot u'(y) = -\int_0^y x \cdot u''(x) dx > 0 \Leftrightarrow u(y) > u'(y) \cdot y$$

□



Proposition

Suppose that bidders are risk-averse and have identical utility functions. With symmetric, independent values, the expected revenue in a first-price auction is greater than in a second-price auction.

Proof

- second-price auction:

  bid your value is still a dominant strategy.

  $\rightarrow$ expected price is the same as with risk-neutrality.

- first-price auction:

Suppose all bidders use the same increasing and differentiable function

$$\gamma : [0, \omega] \rightarrow \mathbb{R}_+$$

with $\gamma(0) = 0$.

It is not optimal to bid $b > \gamma(\omega)$ for any value.

Given value v, the optimization problem is

$$\max_{z \in [0,\omega]} G(z) \cdot u(v - \gamma(z))$$

First order condition:

$$g(z) \cdot u(v - \gamma(z)) - G(z) \cdot u'(v - \gamma(z)) \cdot \gamma'(z) = 0$$

If γ is an equilibrium strategy, then $z = v$ satisfies the FOCs.

$$\Rightarrow g(v) \cdot \frac{u(v - \gamma(v))}{u'(v - \gamma(v))} - G(v) \cdot \gamma'(v) = 0 \tag{5.1}$$

By $\frac{u(x)}{u'(x)} > x \ \forall \ x > 0$:

$$\begin{aligned}
g(y) \cdot (y - \gamma(y)) - G(y) \cdot \gamma'(y) &< 0 \ \forall \ y > 0 \\
\Leftrightarrow g(y) \cdot y &< g(y) \cdot \gamma(y) + G(y) \cdot \gamma'(y) \ \forall \ y > 0 \\
\Rightarrow \int_0^v g(y) \cdot y dy &< \int_0^v (G(y) \cdot \gamma(y))' \, dy \\
&= G(v) \cdot \gamma(v) - G(0) \cdot \gamma(0)
\end{aligned}$$

$$\Rightarrow \ \gamma(v) > \frac{1}{G(v)} \int_0^v g(y) \cdot y dy = \beta(v) \ \forall \ v \in (0, \omega]$$

□

Effect of risk-aversion in first-price auction:
⇒ increase in equilibrium bids
⇒ increase in expected equilibrium revenues

If bidders are risk averse, the expected revenue of a first price auction is higher than the expected revenue of a second price auction!

Example

Suppose $u(x) = x^\theta \ \forall \ x \geq 0$ with $\theta \in (0, 1)$ and $v_i \overset{U,\, iid}{\sim} [0, 1]$ for $i = 1, 2$. Then $G(v) = F(v) = v$ and $g(v) = f(v) = 1$. As $u'(x) = \theta \cdot \frac{u(x)}{x}$ we have

$\frac{u(x)}{u'(x)} = \frac{x}{\theta}$. Then the first order condition is

$$\frac{v - \gamma(v)}{\theta} = v \cdot \gamma'(v) \tag{5.2}$$

Consider $\tilde{F}(v) = v^{\frac{1}{\theta}}$. Then $\tilde{G}(v) = v^{\frac{1}{\theta}}$, $\tilde{g}(v) = \frac{1}{\theta} \cdot v^{\frac{1}{\theta}-1}$.

If $\gamma(v)$ solves (5.2), then $\gamma(v)$ solves (5.1) with $\tilde{G}(v)$ and

$$\gamma(v) = \frac{1}{\tilde{G}(v)} \cdot \int_0^v y \cdot \tilde{g}(y) dy = \frac{1}{1+\theta} \cdot v$$

For a given $\theta \in (0,1)$, the equilibrium strategies of

- N, $F(v) = v$, $u(x) = x^\theta$
- N, $\tilde{F}(v) = v^{\frac{1}{\theta}}$, $\tilde{u}(x) = x$
- $\tilde{N} = \frac{N-1+\theta}{\theta}$, $F(v) = v$, $\tilde{u}(x) = x$

are identical.

5.3 Asymmetries among Bidders

Two risk-neutral bidders, $v_i \sim [0, \omega_i]$ with cdf F_i and $\omega_1 > \omega_2$

Second-Price Auction

Bid your value is dominant.