Übung zu Kapitel 6LS

30.6.2026

#1 $$\text{Bier}_i = \beta_0 + \beta_1 \text{Gehalt}_i + \beta_2 \text{Preis}_i + \beta_3 \text{Bildung}_i + \beta_4 \text{Frau}_i + u_i.$$

B G P E F

$\tilde{U}_i = \frac{U_i}{G_i}$

$$\text{Var}(u_i \mid X) = \sigma^2 \cdot \underbrace{\text{Gehalt}_i^2}_{G_i} = \sigma_i^2 \quad \Big| \quad \frac{\sigma_i^2}{\sigma^2} = G_i^2$$

$$\Omega = \begin{pmatrix} G_1^2 & & & 0 \\ & G_2^2 & & \\ & & \ddots & \\ 0 & & & G_n^2 \end{pmatrix} \qquad P = \begin{pmatrix} \frac{1}{G_1} & & & 0 \\ & \frac{1}{G_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{G_n} \end{pmatrix}$$

$$\tilde{B}_i = \frac{B_i}{G_i} \qquad \tilde{L} = \frac{1}{G_i} \qquad \tilde{G}_i = \frac{G_i}{G_i} = 1 \qquad \tilde{P}_i = \frac{P_i}{G_i} \qquad \tilde{E}_i = \frac{E_i}{G_i} \qquad \tilde{F} = \frac{F_i}{G_i}$$

$$\tilde{v}_i = \frac{v_i}{G_i}$$

$$Var(\tilde{v}_i \mid X) = Var\left(\frac{v_i}{G_i} \mid X\right) = \frac{1}{G_i^2} Var(v_i \mid X)$$

$$= \frac{1}{G_i^2} \sigma^2 \cdot G_i^2$$

$$\Rightarrow \not{\Sigma}(\tilde{v} \mid X) = \sigma^2 \cdot I \quad \text{MLR 5} \checkmark$$

#2

$$\underbrace{\mathrm{Var}(u_i \mid X)}_{\sigma^2 \cdot \Omega} = \begin{pmatrix} \sigma_1^2 I_{n_1} & 0 \\ 0 & \sigma_2^2 I_{n_2} \end{pmatrix}$$

$n = n_1 + n_2$

$$= \begin{pmatrix} \begin{pmatrix} \sigma_1^2 & & 0 \\ & \sigma_1^2 & \\ & \ddots & \\ 0 & & \sigma_1^2 \end{pmatrix}_{n_1 \times n_1} & 0 \\ 0 & \begin{pmatrix} \sigma_2^2 & & 0 \\ & \sigma_2^2 & \\ & \ddots & \\ 0 & & \sigma_2^2 \end{pmatrix}_{n_2 \times n_2} \end{pmatrix}$$

$$\Omega = \begin{pmatrix} \frac{\sigma_1^2}{\sigma^2} \cdot I_{n_1} & 0 \\ 0 & \frac{\sigma_2^2}{\sigma^2} I_{n_2} \end{pmatrix}$$

$$y = X\beta + u$$

$$\Leftrightarrow$$

$$\underbrace{Py}_{\tilde{y}} = \underbrace{PX}_{\tilde{X}}\beta + \underbrace{Pu}_{\tilde{u}}$$

$$\tilde{y}_i = \begin{cases} \frac{\sigma}{\sigma_1} \cdot y_i & \text{falls } i \le n_1 \\ \frac{\sigma}{\sigma_2} \cdot y_i & \text{falls } i > n_1 \end{cases}$$

$$\Omega^{-1} = \begin{pmatrix} \frac{\sigma^2}{\sigma_1^2} I_{n_1} & 0 \\ 0 & \frac{\sigma^2}{\sigma_2^2} I_{n_2} \end{pmatrix}$$

$$\tilde{X}_{ij} = \begin{cases} \frac{\sigma}{\sigma_1} x_{ij} & \text{falls } i \le n_1 \\ \frac{\sigma}{\sigma_2} x_{ij} & \text{falls } i > n_1 \end{cases}$$

$$P = \Omega^{-\frac{1}{2}} = \begin{pmatrix} \frac{\sigma}{\sigma_1} I_{n_1} & 0 \\ 0 & \frac{\sigma}{\sigma_2} I_{n_2} \end{pmatrix}$$

$$\tilde{u}_i = \begin{cases} \frac{\sigma}{\sigma_1} u_i & \text{falls } i \le n_1 \\ \frac{\sigma}{\sigma_2} u_i & \text{falls } i > n_1 \end{cases}$$

$$y = X\beta + u$$

OLS - Schätzer:

$$\hat{\beta} = (X'X)^{-1}X'y$$

Varianz (falls MLR5 erfüllt)

$$\mathbb{V}(\hat{\beta}|X) = \sigma^2 (X'X)^{-1}$$

$$\tilde{y} = \tilde{X}\beta + \tilde{u}$$

GLS - Schätzer

$$\tilde{\beta} = (\tilde{X}'\tilde{X})^{-1}\tilde{X}'\tilde{y}$$

$$= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$$

Varianz (falls MLR5' erfüllt)

$$\mathbb{V}(\tilde{\beta}|X) = \sigma^2(\tilde{X}'\tilde{X})^{-1}$$

$$= \sigma^2(X'\Omega^{-1}X)^{-1}$$

mit $\Omega^{-1} = \begin{pmatrix} \frac{\sigma^2}{\sigma_1^2} I_{n_1} & 0 \\ 0 & \frac{\sigma^2}{\sigma_2^2} I_{n_2} \end{pmatrix}$