Aufgaben zu Kapitel 4

26.5.26

Die Zufallsvariable

$$t = \frac{\hat{\beta}_j - \beta_j}{\sqrt{\widehat{\mathrm{Var}}(\hat{\beta}_j | X)}}$$

ist t-verteilt mit $n-k-1$ Freiheitsgraden.

Im Allgemeinen:

$$X \sim \mathcal{N}(0,1)$$

$$Y \sim \chi^2_{n-k-1}$$

X, Y unabhängig

$$\rightarrow \frac{X}{\sqrt{Y/n-k-1}}$$

t-verteilt mit $n-k-1$ Freiheitsgraden

$$t = \frac{\hat{\beta}_j - \beta_j}{\sqrt{\widehat{Var}(\hat{\beta}_j \mid X)}} = \frac{\boxed{\dfrac{\hat{\beta}_j - \beta_j}{\sqrt{Var(\beta_j \mid X)}}}}{\boxed{\dfrac{\sqrt{\widehat{Var}(\beta_j \mid X)}}{\sqrt{Var(\beta_j \mid X)}}}}$$

$$\sim \mathcal{N}(0,1) \quad \text{Aufgabe 1}$$

$$= \text{"}\sqrt{\chi / n-k-1}\text{"} \quad \text{Aufgabe 2}$$

1

$$v \sim \mathcal{N}(\mu, \sigma_v^2)$$

$$\frac{\hat{\beta}_j - \beta_j}{\sqrt{Var(\beta_j)}} \sim \mathcal{N}(0,1)$$

zufällig, fest, fest

$$z = \frac{v - \mu}{\sqrt{\sigma_v^2}} \sim \mathcal{N}\left(E[z], Var(z)\right) \quad \leadsto z \sim \mathcal{N}(0,1)$$

$$E[z] = E\left[\frac{v-\mu}{\sqrt{\sigma_v^2}}\right] = \frac{1}{\sqrt{\sigma_v^2}} E[v-\mu] = \frac{1}{\sqrt{\sigma_v^2}}\left(\underbrace{E[v]}_{=\mu} - \mu\right) = 0$$

$$Var(z) = Var\left(\frac{v-\mu}{\sqrt{\sigma_v^2}}\right) = \frac{1}{(\sqrt{\sigma_v^2})^2} \cdot Var(v-\mu) = \frac{1}{\sigma_v^2} \underbrace{Var(v)} = \frac{1}{\sigma_v^2} \cdot \sigma_v^2 = 1$$

$$\frac{1}{\sigma} u \sim \mathcal{N}(0, I_n)$$

$$E\left[\frac{u}{\sigma}\right] \qquad \Sigma\left(\frac{u}{\sigma}\right)$$

MLR 6: $u \sim \mathcal{N}(0, \sigma^2 I)$

$$E\left[\frac{1}{\sigma} \cdot u\right] = \frac{1}{\sigma} \cdot E[u] = \frac{1}{\sigma} \cdot 0 = 0 \checkmark$$

$$\Sigma\left(\frac{1}{\sigma} \cdot u\right) = \left(\frac{1}{\sigma}\right)^2 \cdot \Sigma(u) = \frac{1}{\sigma^2} \cdot \sigma^2 \cdot I = I \checkmark$$

2

$$M_x = I - X(X'X)^{-1}X'$$

$$\hat{u} = y - \hat{y} = y - X\hat{\beta} = y - X(X'X)^{-1}X'y$$

$$= \underbrace{[I - X(X'X)^{-1}X']}_{M_x} y$$

$$\hat{u} = M_x \cdot y = \dots = M_x \cdot u$$

idempotent
symmetrisch

$$y = X\beta + u$$

$$\hat{u}'\hat{u} = (M_x u)' M_x u = u' \underbrace{M_x' M_x}_{M_x} \cdot u = u' M_x u$$

$$\hat{\sigma}^2 = \frac{1}{n-k-1} \cdot \overbrace{\hat{u}'\hat{u}}^{SSR}$$

$$(n-k-1)\frac{\hat{\sigma}^2}{\sigma^2} = u' M_x u \frac{1}{\sigma^2}$$

$$= \left(\frac{u}{\sigma}\right)' M_x \frac{u}{\sigma}$$

$$\left(\frac{u}{\sigma}\right)' \left(\frac{u}{\sigma}\right) = \sum_{i=1}^{n} \left(\frac{u_i}{\sigma}\right)^2 \sim \chi^2_n$$

$$\left(\frac{u}{\sigma}\right)' M_x \frac{u}{\sigma} \sim \chi^2_{n-k-1}$$

$\uparrow$

idempotent symmetrisch

$$rk(M_x) = n-k-1 = tr(M_x)$$

$$\frac{1}{n-k-1} \cdot \hat{u}'\hat{u}$$

Also $$\left(\frac{\hat{u}}{\sigma}\right)' \frac{\hat{u}}{\sigma} = \frac{1}{\sigma^2} \cdot \hat{u}'\hat{u} = \frac{1}{\sigma^2} \cdot (n-k-1) \cdot \boxed{\hat{\sigma}^2}$$

$$(n-k-1) \frac{\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-k-1}$$

3

$$\frac{\sqrt{\widehat{Var}(\hat{\beta}_j \mid X)}}{\sqrt{Var(\hat{\beta}_j \mid X)}} = \frac{\sqrt{\hat{\sigma}^2 (X'X)^{-1}_{j,j}}}{\sqrt{\sigma^2 (X'X)^{-1}_{j,j}}} = \frac{\sqrt{\hat{\sigma}^2}}{\sqrt{\sigma^2}}$$

$$= \sqrt{\frac{\hat{\sigma}^2}{\sigma^2}} = \sqrt{\underbrace{(n-k-1)\frac{\hat{\sigma}^2}{\sigma^2}}_{\sim \chi^2_{n-k-1}} \Big/ (n-k-1)}$$

$$t_j = \frac{\hat{\beta}_j - \beta_j}{\sqrt{\widehat{\operatorname{Var}}(\hat{\beta}_j \mid X)}} = \frac{\overbrace{\dfrac{\hat{\beta}_j - \beta_j}{\sqrt{\operatorname{Var}(\hat{\beta}_j \mid X)}}}^{x \;\sim\; \mathcal{N}(0,1)}}{\sqrt{\underbrace{\dfrac{(n-k-1)\,\hat{\sigma}^2}{\sigma^2}}_{y \;\sim\; \chi^2_{n-k-1}} \Big/ (n-k-1)}}$$

Als letztes wäre zu zeigen: x & y unabhängig

4

$k+1=4$ $\quad$ $k=3$

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Modell 1: KQ, benutze die Beobachtungen 1-209
Abhängige Variable: lsalary

              Koeffizient    Std.-fehler    t-Quotient    p-Wert
  -------------------------------------------------------------
  const       4,31171        0,315433        13,67        1,16e-30 ***
  lsales      0,280315       0,0353200        7,936       1,34e-13 ***
  roe         0,0174168      0,00409230       4,256       3,17e-05 ***
  ros         0,000241655    0,000541802      0,4460      0,6561

Mittel abhängige Var.    6,950386   Stdabw. abhängige Var.   0,566374
Summe quad. Residuen     47,86083   Stdfehler Regression     0,483185
R-Quadrat                0,282685   Korrigiertes R-Quadrat   0,272188
F(3, 205)                26,92930   P-Wert(F)                1,00e-14
Log-Likelihood          -142,5213   Akaike-Kriterium         293,0426
Schwarz-Kriterium        306,4119   Hannan-Quinn-Kriterium   298,4479

Abgesehen von Konstante war p-Wert am höchsten für Variable 6 (ros)
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$\hat\beta_0$ const, $\hat\beta_1$ lsales, $\hat\beta_2$ roe, $\hat\beta_3$ ros

Test-Entsch.

$|t_3| < c$

$\Rightarrow H_0: \beta_3 = 0$ nicht ablehnen

ros $\nearrow$ 50

$\rightarrow$ salary $\nearrow \hat\beta_3$ %

$0.00024 \cdot 50$

≈ 0.012 %

$$H_0: \beta_3 = 0 \Rightarrow \hat\beta_3 \sim N(0, Var(\hat\beta_3 | X))$$

$$t_3 = \frac{\hat\beta_3 - 0}{\underbrace{\sqrt{\widehat{Var}(\hat\beta_3)}}_{se(\hat\beta_3)}} \sim t_{205}$$

$= 0,4460$

Fehlerw'keit: α

suche c sodass

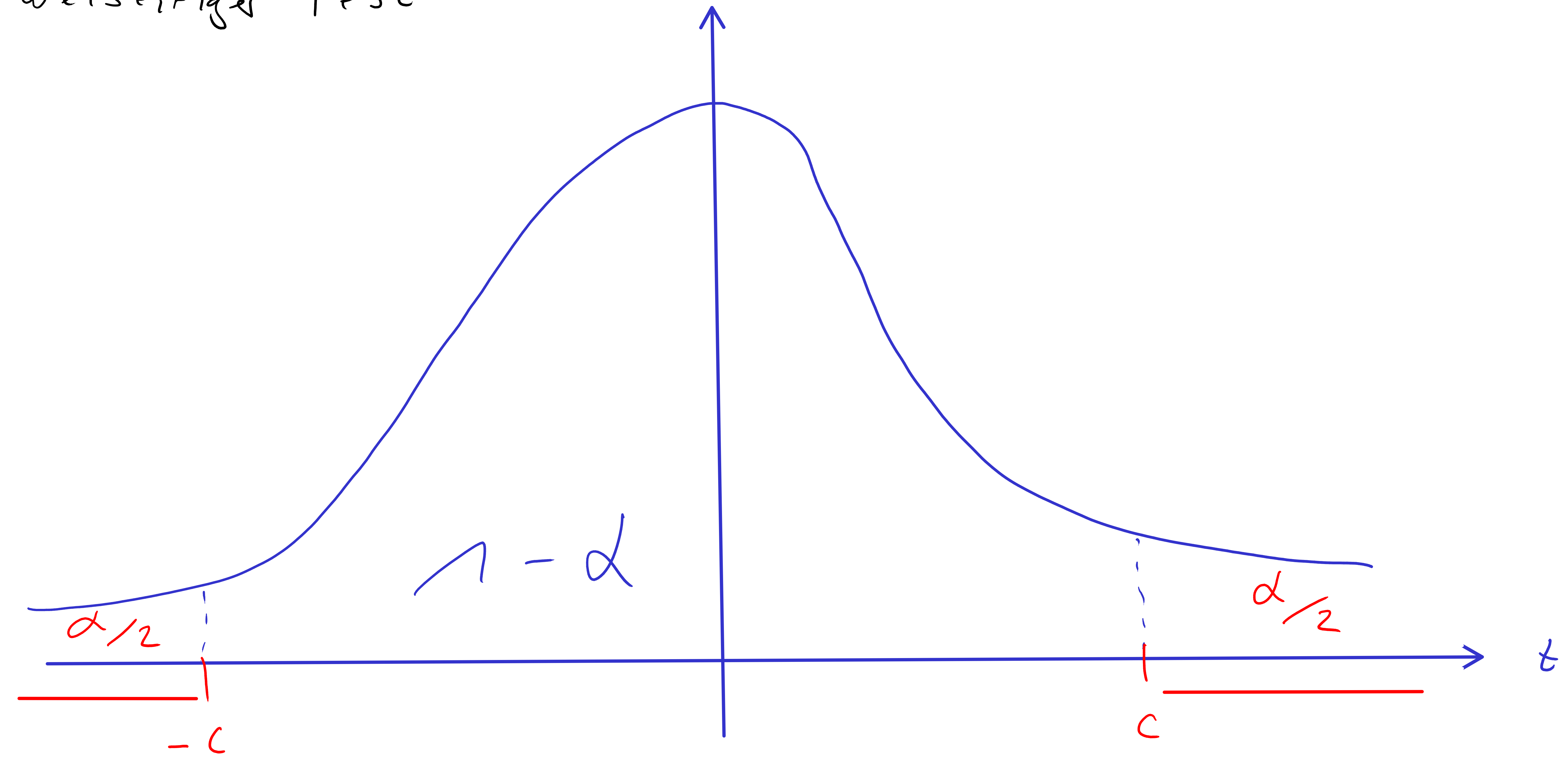
$c_{t_{120};\, 99,5\%} = 2,617$

$$Prob(t_3 \geq c) = \frac{\alpha}{2}$$

$$\Leftrightarrow Prob(t_3 < -c) = \frac{\alpha}{2} \Leftrightarrow Prob(-c \leq t_3 \leq c) = 1-\alpha$$

$$1{,}16 \, e - 30 = 1{,}16 \cdot 10^{-30}$$

Zweiseitiger Test



$$1-\alpha+\frac{\alpha}{2}=1-\frac{\alpha}{2}$$

$$\mathrm{Prob}(t \leq c)=1-\frac{\alpha}{2}$$

noch # 4

Einseitiger test

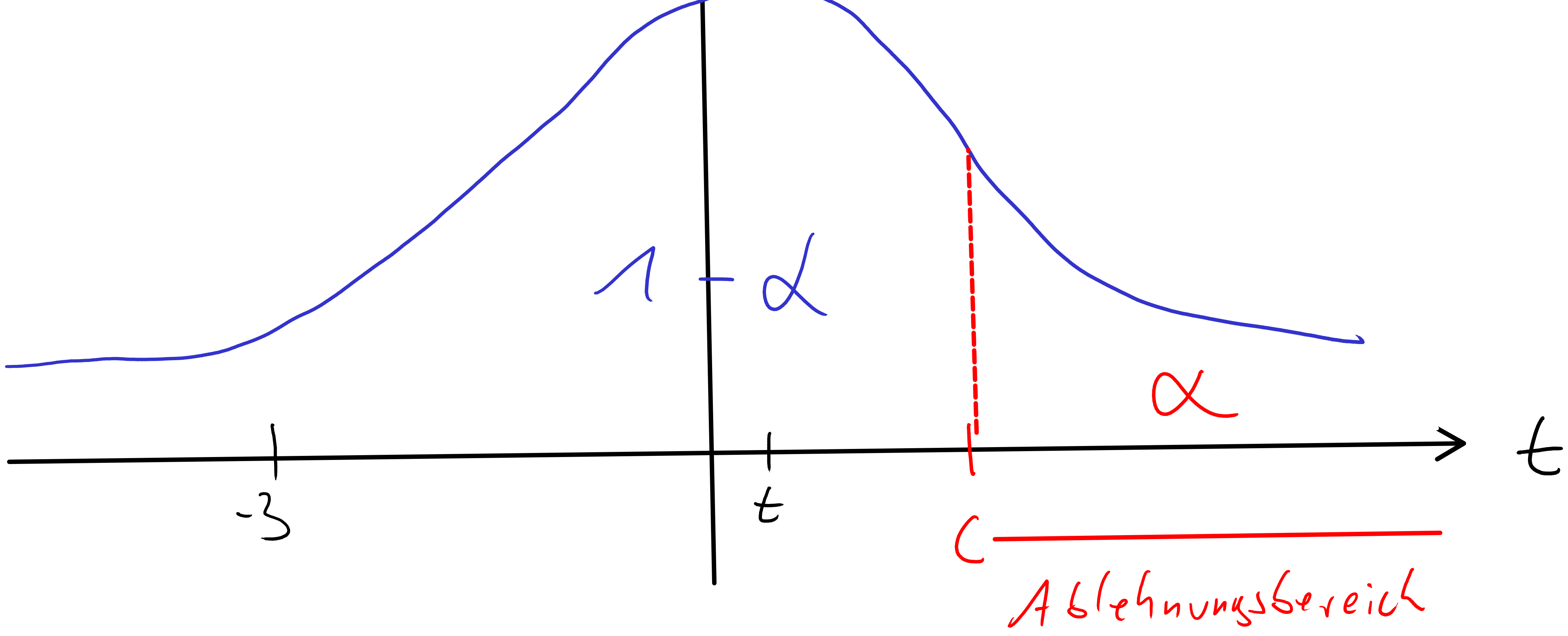
$\tilde{H}_0$ $\beta_3 \leq 0$ $\tilde{H}_1$: $\beta_3 > 0$

$c_{t_{120};\, 99\%} = 2{,}358$

$$t_4 = \frac{\hat{\beta}_3 - 0}{se(\hat{\beta}_3)} = 0.44$$

$t_4 < c \to$ H_0 nicht verwerfen

$Prob(t \leq c) = 1 - \alpha$



5

Jahr:	2007	2008	2009	2010	2011
Zeit:	10,03s	9,69s	9,58s	9,82s	9,76s
Jahr:	**2012**	**2013**	**2014**	**2015**	**2016**
Zeit:	9,63s	9,77s	9,98s	9,79s	9,81s

$$y_i = \beta_0 + u_i$$

$$\hat{\beta}_0 = \bar{y} \qquad X = \iota$$

$$= (\iota'\iota)^{-1}\iota' y = (n)^{-1}\sum y_i = \bar{y}$$

$$Var(\hat{\beta}_0) = \sigma^2 \cdot (\iota'\iota)^{-1} = \frac{1}{n}\sigma^2$$

$$\widehat{Var}(\hat{\beta}_0) = \frac{1}{n}\hat{\sigma}^2$$

$$\hat{u} = y - \hat{\bar{y}} = y - \hat{\beta}_0 = y - \bar{y}$$

$$\hat{\sigma}^2 = \frac{1}{n-k-1}\hat{u}'\hat{u}$$

$k=0$

$$\hat{u}'\hat{u} = \sum_{i=1}^{n}(y_i - \bar{y})^2$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

$$t_0 = \frac{\bar{y} - \gamma}{\sqrt{\frac{1}{n}\frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

5

$\gamma = 9{,}75$

```
Modell 1: KQ, benutze die Beobachtungen 1-10
Abhängige Variable: best

             Koeffizient   Std.-fehler   t-Quotient    p-Wert
  -----------------------------------------------------------------
  const        9,78600      0,0442016       221,4      3,98e-18 ***

Mittel abhängige Var.    9,786000   Stdabw. abhängige Var.   0,139778
Summe quad. Residuen     0,175840   Stdfehler Regression     0,139778
R-Quadrat                0,000000   Korrigiertes R-Quadrat   0,000000
Log-Likelihood           6,014444   Akaike-Kriterium        −10,02889
Schwarz-Kriterium       −9,726303   Hannan-Quinn-Kriterium  −10,36082
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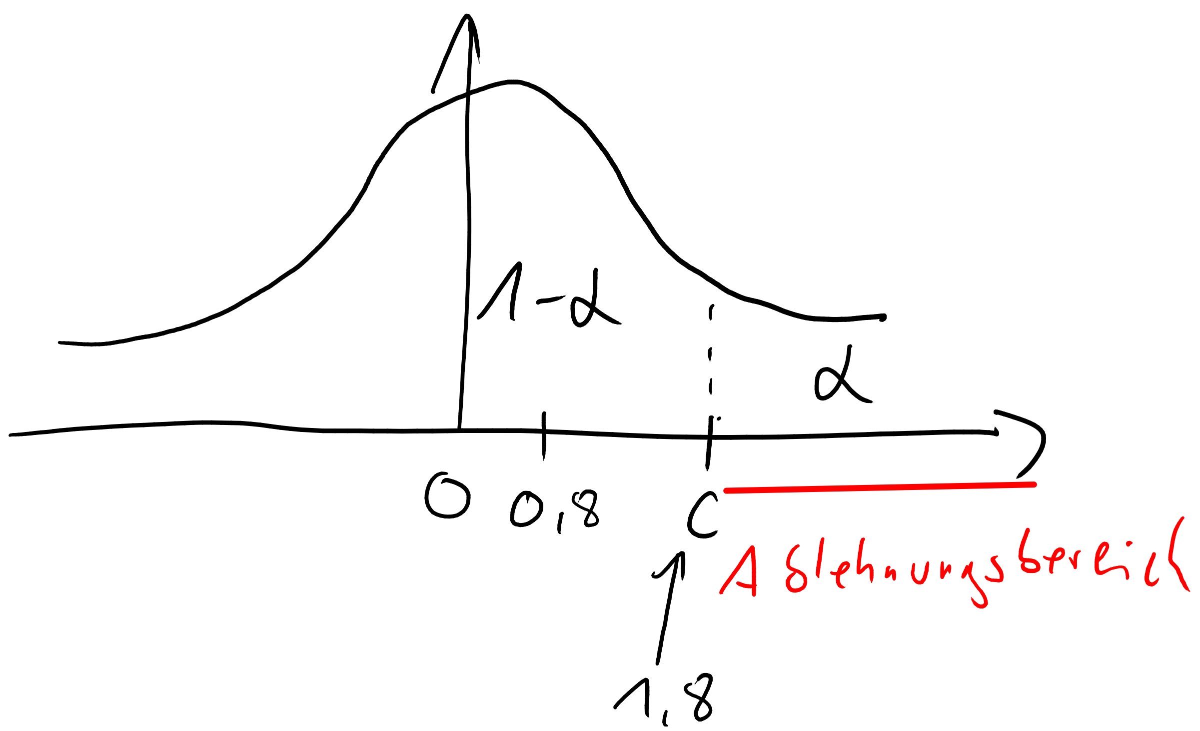
$H_0: \beta_0 \leq 9{,}75$

$H_1: \beta_0 > 9{,}75$

aus Aufgabenstellung

$$t_0 = \frac{9{,}786 - 9{,}75}{0{,}0442016} = \frac{0.036}{0.0442016} = 0.81445$$



$C_{t_9;\,95\%} = 1{,}833$

$t_0 < C$

$\Rightarrow$ H_0 nicht ablehnen.