Blatt 4

Aufgabe 1

i	y_i	x_i	z_i
1	2	1	0
2	3	2	1
3	5	3	2

a)

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

$$\det(X) = 4 + 1 + 0 - 0 - 3 - 2 = 5 - 5 = 0$$

b)

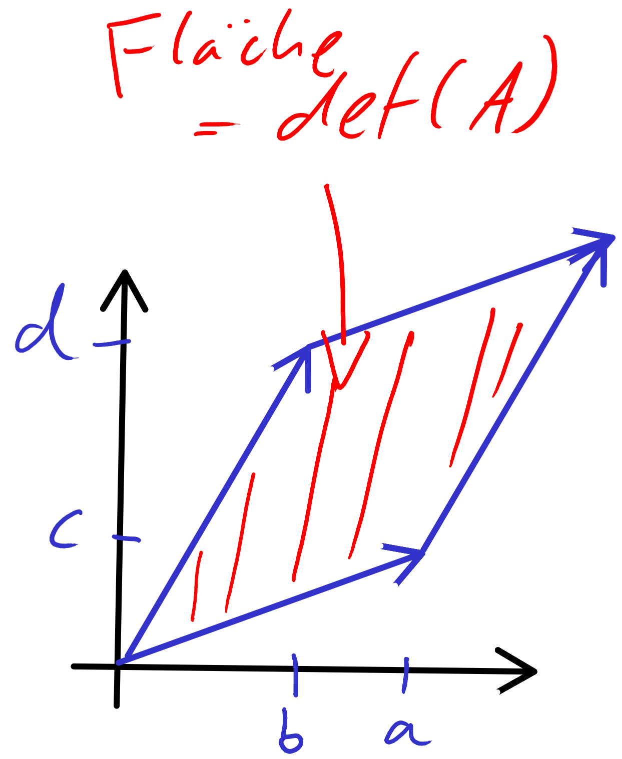
$$X = (\iota, x, z) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{matrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{matrix}$$

$$rk(X) = 2 < 3$$

es gilt $\iota + z = x$

MLR 3 verletzt.

Determinante einer Matrix



$$A = a \in \mathbb{R}^{1\times 1} \Rightarrow \det(A) = a$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2} \Rightarrow \det(A) = a\cdot d - b\cdot c$$

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix} \in \mathbb{R}^{3\times 3}$$

$$\det(A) = aei + bfg + cdh - gec - hfa - idb$$

c) $y = X\beta + u$

$$\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \beta_0 + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \beta_1 + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \beta_2 + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

d) $y \in \mathbb{R}^3$ $\quad X \in \mathbb{R}^{3 \times 3}$ $\quad \beta \in \mathbb{R}^3$ $\quad u \in \mathbb{R}^3$

Aufgabe 2 $\quad \bar{y} = 10, \bar{x} = 2, \bar{z} = 3, s_{xx} = 4, s_{zz} = 9, s_{xz} = 3, s_{xy} = 8, s_{zy} = 12$

$$X = \begin{pmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & z_n \end{pmatrix} \qquad X'X = n \begin{pmatrix} 1 & \bar{x} & \bar{z} \\ \bar{x} & \overline{xx} & \overline{xz} \\ \bar{z} & \overline{xz} & \overline{zz} \end{pmatrix}$$

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \qquad \overline{xx} = \frac{1}{n} \sum_{i=1}^{n} x_i \cdot x_i \qquad \overline{xz} = \frac{1}{n} \sum_{i=1}^{n} x_i z_i$$

$$s_{xx} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x}) \qquad s_{xz} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(z_i - \bar{z})$$

$$= \frac{1}{n} \sum_{i=1}^{n} x_i \cdot x_i - (\bar{x})^2$$

$$\overline{zz} = s_{zz} + \bar{z}^2 = 9 + 3^2 = 18$$

$$\Leftrightarrow \overline{xx} = s_{xx} + (\bar{x})^2 = 4 + 2^2 = 8$$

$$\overline{xz} = s_{xz} + \bar{x} \cdot \bar{z} = 3 + 2 \cdot 3 = 9$$

1. Momente

$$X'X = n \begin{pmatrix} 1 & \overset{\bar{x}}{2} & 3 \\ 2 & \overset{\overline{xx}}{8} & 9 \\ 3 & 9 & 18 \end{pmatrix}$$

2. Momente

zum Beispiel

$\overline{xx} - \bar{x}\cdot\bar{x} = 8 - 2\cdot 2 = 4 = S_{xx}$

$X'X$ positiv definit symmetrisch? ✓

Def. pos. def: $v'X'X\cdot v > 0$ für alle $v \in \mathbb{R}^3$

Berechne „führenden Hauptminoren":

$\det(1) = 1 > 0$ ✓

$\det\begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix} = 1\cdot 8 - 2\cdot 2 = 8 - 4 = 4 > 0$ ✓

$$\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ 2 & 8 & 9 & 2 & 8 \\ 3 & 9 & 18 & 3 & 9 \end{array}$$

$$\det(X'X) = 1 \cdot 8 \cdot 18 + 2 \cdot 9 \cdot 3 + 3 \cdot 2 \cdot 9$$
$$- 3 \cdot 8 \cdot 3 - 9 \cdot 9 \cdot 1 - 18 \cdot 2 \cdot 2$$

$$= 9(16 + 6 + 6 - 8 - 9 - 8)$$

$$= 9(22 - 2 - 17) = 9 \cdot 3 = 27 > 0$$

Alle führenden Hauptminoren sind positiv, deshalb ist $X'X$ positiv definit.

b)

$$(X'X)^{-1} = \frac{1}{9} \begin{pmatrix} 21 & -3 & -3 \\ -3 & 3 & -1 \\ -2 & -1 & \frac{4}{3} \end{pmatrix}$$

$$(X'X)^{-1} \cdot X'X = \frac{1}{9} \begin{pmatrix} 21 & -3 & -2 \\ -3 & 3 & -1 \\ -2 & -1 & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 8 & 9 \\ 3 & 9 & 18 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} \cdot & \begin{array}{l} 21 \cdot 2 + (-3) \cdot 8 + (-2) 9 \\ = 42 - 24 - 18 = 0 \end{array} & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \begin{array}{l} -2 \cdot 3 + (-1) \cdot 9 + \frac{4}{3} \cdot 18 \\ = -6 - 9 + 24 = 9 \end{array} \end{pmatrix}$$

c) $$X'y = \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ z_1 & z_2 & \dots & z_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i \cdot y_i \\ \sum_{i=1}^n z_i \cdot y_i \end{pmatrix}$$

$3 \times \underline{n}$ $\quad$ $\underline{n} \times 1$

$$X = \begin{pmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & z_n \end{pmatrix}$$

$$\to \quad n \begin{pmatrix} \bar{y} \\ \overline{xy} \\ \overline{zy} \end{pmatrix} = n \begin{pmatrix} 10 \\ 28 \\ 42 \end{pmatrix} = X'y$$

$$S_{xy} = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) = \overline{xy} - \bar{x} \cdot \bar{y} \iff \overline{xy} = \underset{8}{S_{xy}} + \underset{2 \cdot 10}{\bar{x} \cdot \bar{y}} = 28$$

$$S_{zy} = \dots = \overline{zy} - \bar{z} \cdot \bar{y} \iff \overline{zy} = \underset{12}{S_{zy}} + \underset{3 \cdot 10}{\bar{z} \cdot \bar{y}} = 42$$

d) $\hat{\beta} = (X'X)^{-1}X'y$

$$\cancel{(X'X)^{-1}} = \frac{1}{9}\frac{1}{n}\begin{pmatrix} 21 & -3 & -\cancel{3}^{2} \\ -3 & 3 & -1 \\ -2 & -1 & \frac{4}{3} \end{pmatrix} n \begin{pmatrix} 10 \\ 28 \\ 42 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 21\cdot 10 - 3\cdot 28 - \cancel{3}^{2}\cdot 42 \\ -3\cdot 10 + 3\cdot 28 - 42 \\ -2\cdot 10 - 1\cdot 28 + \frac{4}{3}42 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 210 - 84 - 84 \\ -30 + 84 - 42 \\ -20 - 28 + 56 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 42 \\ 12 \\ 8 \end{pmatrix}$$

3)

a) $(X'X)' = X'(X')' = X'X$ ✓ (Symmetrie)

b) $v' \underbrace{X'X}_{k+1 \times k+1} v \geq 0$ für alle $v \in \mathbb{R}^{k+1}$

$$\underbrace{v'X'}_{(Xv)'} Xv = \underbrace{(Xv)'}_{z'} \underbrace{Xv}_{z} = z' \cdot z = \sum_{i=1}^{k+1} \underbrace{z_i^2}_{\geq 0} \geq 0$$

c) MLR3 $rk(X) = k+1$

alle $k+1$ Spalten von X sind linear unabhängig

$$\sum_{j=1}^{k+1} x_{\cdot j} \cdot \lambda_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \lambda_j = 0 \text{ für } j = 1, \dots, k+1$$

$X'X$ ist positiv definit, falls

$v'X'Xv > 0$ für alle $v \in \mathbb{R}^{k+1}$, $v \neq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$$rk(X) = k+1 \Rightarrow \left[v \neq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \underbrace{X \cdot v}_{\sum_{j=1}^{k+1} x_{\cdot j} \cdot v_j} \neq 0 \right]$$

$$v \neq 0 \Rightarrow z = X \cdot v \neq 0 \Rightarrow z' \cdot z > 0$$