Übung 3

Aufgabe 1)

$$A = \begin{pmatrix} 2 & 1 \\ -3 & -1 \\ 1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & 12 \\ -7 & -6 \\ 4 & -3 \end{pmatrix}$$

a) $$\underset{3\times 2}{A} + \underset{3\times 2}{B} = \begin{pmatrix} 2+4 & 1+12 \\ -3-7 & -1-6 \\ 1+4 & 3-3 \end{pmatrix} = \underset{3\times 2}{\begin{pmatrix} 6 & 13 \\ -10 & -7 \\ 5 & 0 \end{pmatrix}}$$

$$\underset{3\times 2}{A} - \underset{3\times 2}{B} = \begin{pmatrix} 2-4 & 1-12 \\ -3+7 & -1+6 \\ 1-4 & 3+3 \end{pmatrix} = \underset{3\times 2}{\begin{pmatrix} -2 & -11 \\ 4 & 5 \\ -3 & 6 \end{pmatrix}}$$

b)

$$A = \begin{pmatrix} 2 & 1 \\ -3 & -1 \\ 1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & 12 \\ -7 & -6 \\ 4 & -3 \end{pmatrix}$$

3×2 $\qquad$ 3×2

$\underset{2\times 3}{A'} + \underset{3\times 2}{B}$ ↯

$\underset{3\times 2}{A} + \underset{2\times 3}{B'}$ ↯

c)

$\underset{3\times \underline{2}}{A} \cdot \underset{\underline{3}\times 2}{B}$ ↯

$\underset{3\times \underline{2}}{B} \cdot \underset{\underline{3}\times 2}{A}$ ↯

$$A = \begin{pmatrix} 2 & 1 \\ -3 & -1 \\ 1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & 12 \\ -7 & -6 \\ 4 & -3 \end{pmatrix}$$

$$\underset{3\times 2}{} \qquad\qquad \underset{3\times 2}{}$$

d) $$\underbrace{\underset{2\times 3}{A'} \;\underset{3\times 2}{B}}_{2\times 2} = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 4 & 12 \\ -7 & -6 \\ 4 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2\cdot 4 + -3\cdot(-7) + 1\cdot 4 & 2\cdot 12 + -3\cdot(-6) + 1\cdot(-3) \\ 1\cdot 4 + -1(-7) + 3\cdot 4 & 1\cdot 12 + -1(-6) + 3\cdot(-3) \end{pmatrix}$$

$$= \begin{pmatrix} 33 & 39 \\ 23 & 9 \end{pmatrix}$$

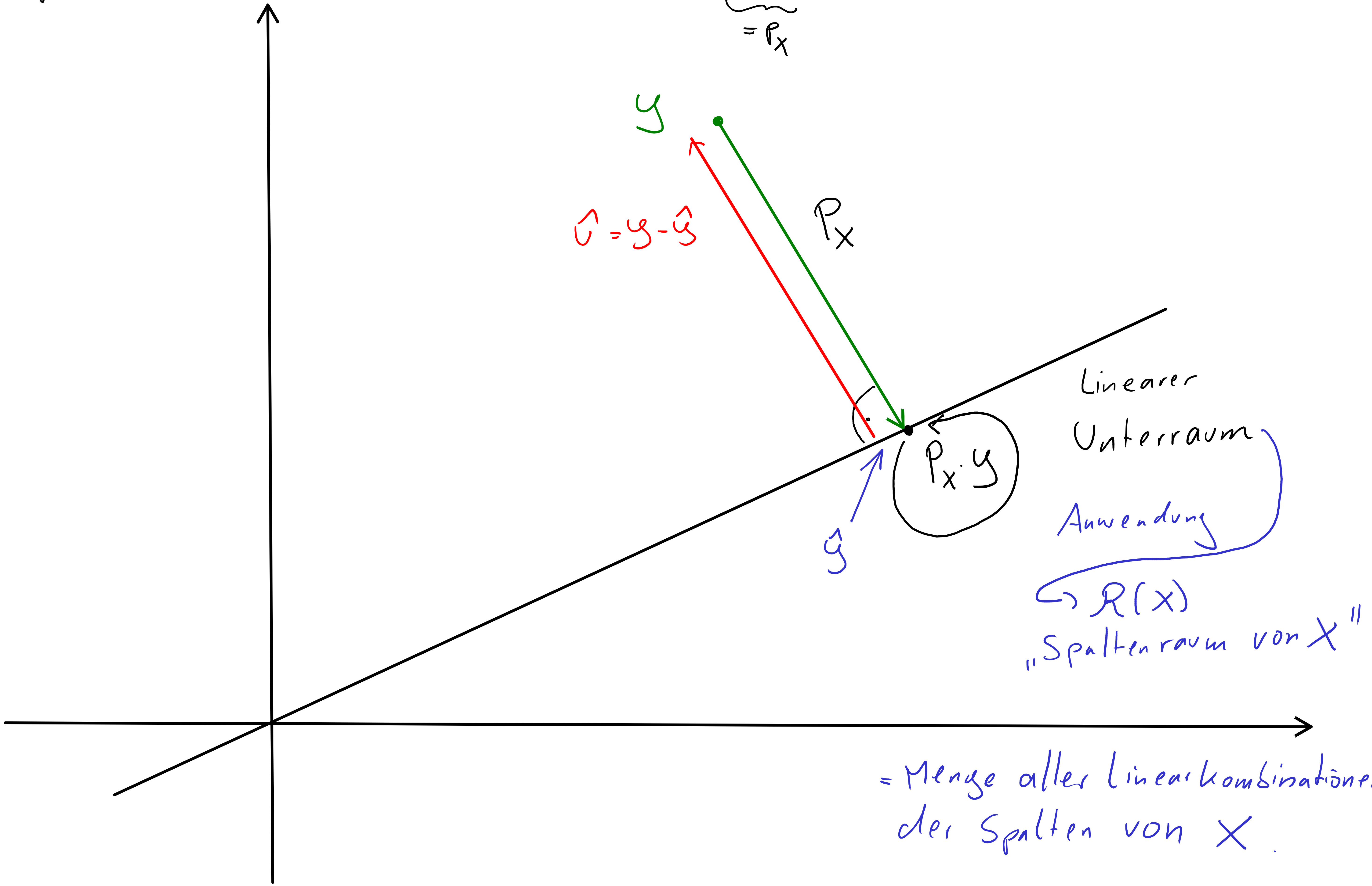
$$A = \begin{pmatrix} 2 & 1 \\ -3 & -1 \\ 1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & 12 \\ -7 & -6 \\ 4 & -3 \end{pmatrix}$$

3×2

$$\underbrace{\underset{3\times 2}{A} \cdot \underset{2\times 3}{B'}}_{3\times 3} = \begin{pmatrix} \circ & \circ & \circ \\ \circ & \begin{matrix} -3\cdot(-7)+(-1)\cdot(-6) \\ =27 \end{matrix} & \circ \\ \circ & \circ & \circ \end{pmatrix}$$

Aufgabe 2

$$\underbrace{P_X P_X}_{=P_X} \cdot y = P_X y \qquad \mathbb{R}^n$$



= Menge aller Linearkombinationen der Spalten von X.

$$P_X = \underbrace{\underbrace{\underset{n\times k+1}{X}\,\underbrace{(\underset{k+1\times n}{X'}\,\underset{n\times k+1}{X})^{-1}}_{k+1\times k+1}}_{n\times k+1}\,\underset{k+1\times n}{X'}}_{n\times n}$$

$$(A^{-1})' = (A')^{-1}$$

$$P_X' = \left[X\,(X'X)^{-1}\,X'\right]'$$

$$(X'X)' = X'X$$

$$= [X']'\,\left[(X'X)^{-1}\right]'\,[X]'$$

$$= X\left[\underbrace{(X'X)'}_{X'X}\right]^{-1}X' = P_X$$

$$P_X \cdot P_X = \underbrace{X(X'X)^{-1}}_{P_X} \underbrace{X' X (X'X)^{-1}}_{I} X'$$

$$= X(X'X)^{-1} I \cdot X'$$

$$= X(X'X)^{-1} X' = P_X$$

$A \cdot A = A$: A ist idempotent.

b) zu zeigen: $P_X = X(X'X)^{-1}X'$

ist positiv semi definit

$\Leftrightarrow \quad v' \underset{n \times n}{P_X} v \geq 0 \quad$ für alle $v \in \mathbb{R}^n$

$P_X \cdot P_X = P_X$

$\Leftrightarrow \quad v' P_X \cdot P_X v \geq 0$

$P_X' = P_X$

$\Leftrightarrow \quad v' P_X' P_X v \geq 0$

$\Leftrightarrow \quad \underbrace{(P_X v)'}_{z'} \underbrace{P_X v}_{\substack{n \times n \; n \times 1 \\ n \times 1 \\ =: z}} \geq 0 \Leftrightarrow z' \cdot z \geq 0$

$\Leftrightarrow \sum_{i=1}^{n} \underbrace{z_i^2}_{\geq 0} \geq 0$

c) z.z. $Z = X\Gamma$

$\Rightarrow P_X \cdot Z = Z$

$$P_X \cdot Z = X \underbrace{(X'X)^{-1} X' \cdot X}_{I} \Gamma = X \cdot I \cdot \Gamma = X\Gamma = Z$$

$$\Gamma = \underset{k+1 \times 1}{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}$$

$$X\Gamma = X_{\cdot 1} \cdot 1 + X_{\cdot 2} \cdot 1 + \dots + X_{\cdot k+1} \cdot 1$$

Linearkombination der Spalten

$\Rightarrow X\Gamma$ liegt im $\underbrace{\text{Spaltenraum von } X}_{R(X)}$

d) $$P_X = X(X'X)^{-1}X'$$

für $X = \iota = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

$$P_X = \underbrace{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}_{X} \left(\underbrace{(1, 1, \dots, 1)}_{X'} \cdot \underbrace{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}_{X} \right)^{-1} \underbrace{(1, 1, \dots, 1)}_{X'}$$

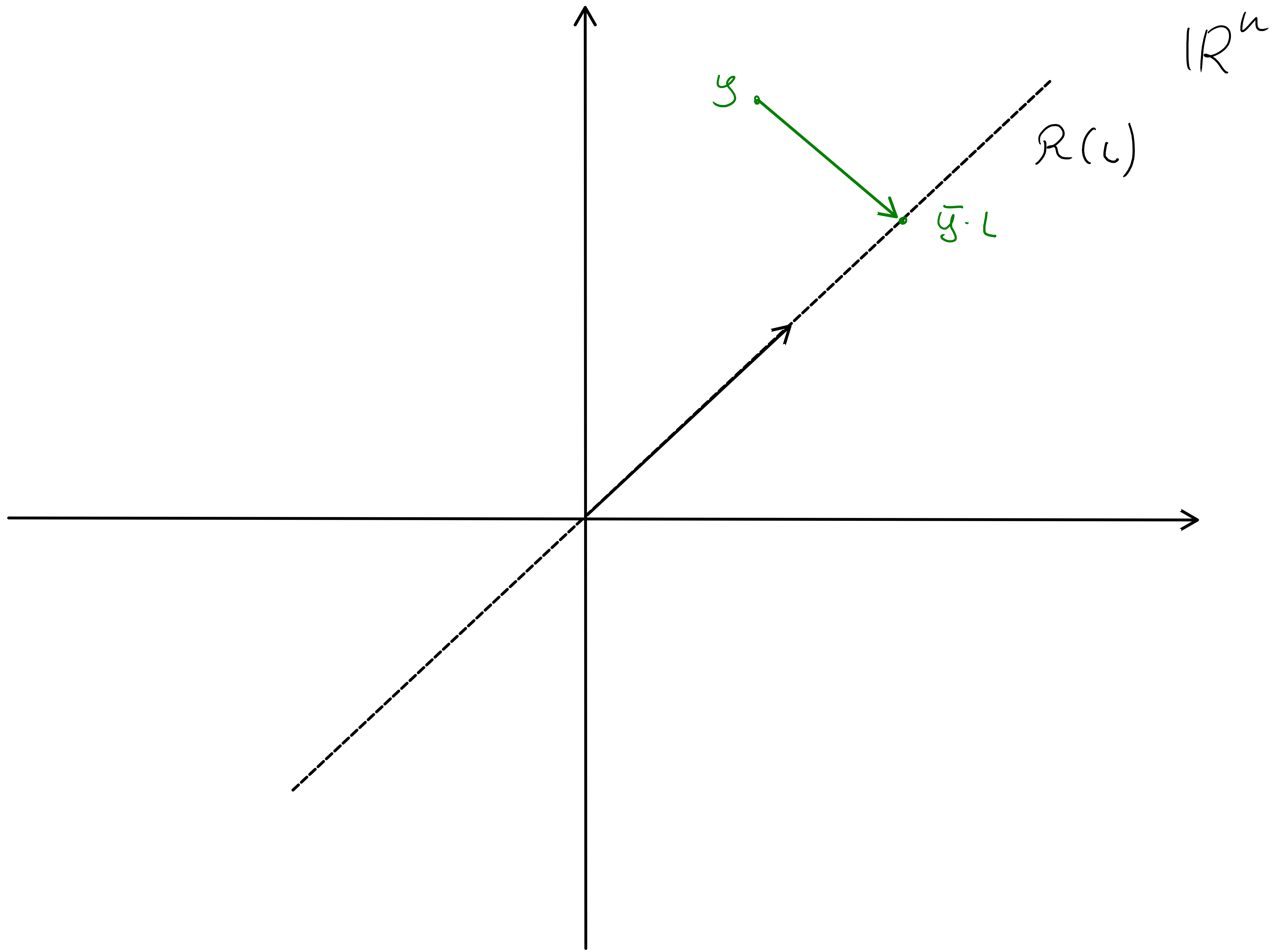
$$= \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \left(\underbrace{\underbrace{1^2 + 1^2 + \dots + 1^2}_{n\text{-mal}}}_{= n} \right)^{-1} (1, 1, \dots, 1) = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} n^{-1} (1, 1, \dots, 1)$$

$$= \frac{1}{n} \underbrace{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}_{n \times 1} \underbrace{(1, 1, \dots, 1)}_{1 \times n} = \frac{1}{n} \begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix}$$

$$P_X = X(X'X)^{-1}X' \quad , \quad X = \iota$$

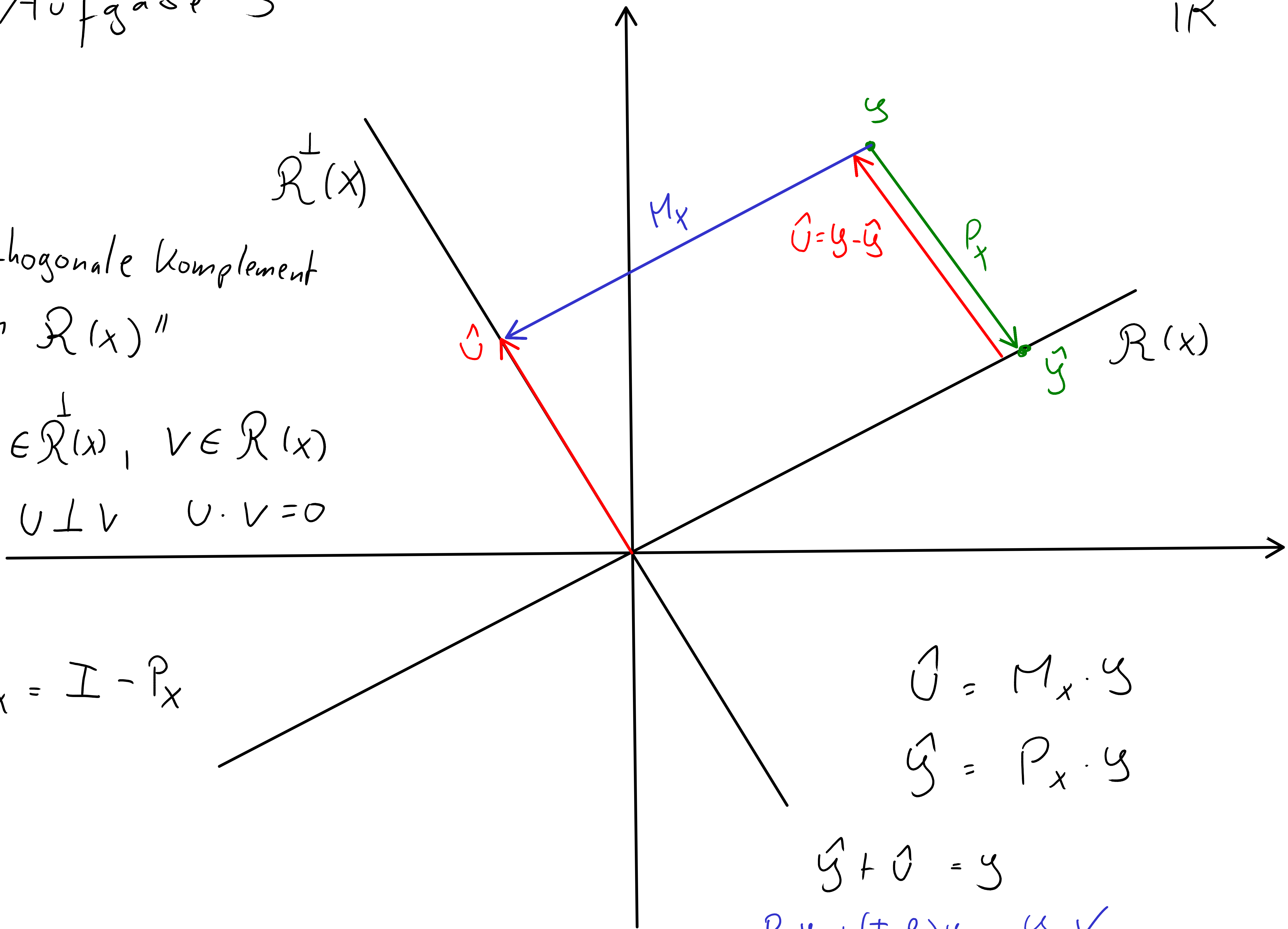
$$P_X \cdot y = \frac{1}{n}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \frac{1}{n}\begin{pmatrix} y_1 \cdot 1 + y_2 \cdot 1 + y_3 \cdot 1 + \cdots + y_n \cdot 1 \\ \sum_{i=1}^{n} y_i \\ \vdots \\ \sum_{i=1}^{n} y_i \end{pmatrix}$$

$$= \begin{pmatrix} \bar{y} \\ \bar{y} \\ \vdots \\ \bar{y} \end{pmatrix}$$


$\mathbb{R}^n$
y
$\mathcal{R}(v)$
$\bar{y} \cdot v$

Aufgabe 3

$\mathbb{R}^n$



„orthogonale Komplement von $R(x)$"

$u \in R^{\perp}(x)$, $v \in R(x)$

$u \perp v \quad u \cdot v = 0$

$M_x = I - P_x$

$\hat{u} = M_x \cdot y$

$\hat{y} = P_x \cdot y$

$\hat{y} + \hat{u} = y$

$P_x y + (I - P_x) y = y$ ✓

a) zu zeigen:

$$\underline{M_x{}' = M_x}$$

$$(I - P_x)' = I' - P_x' = I - P_x = M_x$$

z.z.

$$\underline{M_x \cdot M_x = M_x}$$

$$(I - P_x)(I - P_x) = I \cdot I + I(-P_x) + (-P_x)I + (-P_x)(-P_x)$$

$$= I - P_x - P_x + \underbrace{P_x P_x}_{P_x}$$

$$= I - P_x = M_x$$

b) z.z. $v' M_x \cdot v \geq 0$ für alle v

$$v' \underbrace{M_x M_x}_{= \text{ / idempotent}} \cdot v \geq 0$$

symmetrisch

$$v' M_x' M_x \cdot v \geq 0$$

$$\underbrace{(M_x \cdot v)'}_{z'} \underbrace{M_x \cdot v}_{z} \geq 0$$

$$z' \cdot z \geq 0 \iff \sum_{i=1}^{n} \underbrace{z_i^2}_{\geq 0} \geq 0$$

c) zZ $M_X \cdot X = 0$

$$(I - P_X) \cdot X = \underbrace{I \cdot X}_{= X} - \underbrace{P_X \cdot X}_{= X} = X - X = 0$$

$$M_X \cdot P_X = (I - P_X) P_X = I \cdot P_X - P_X P_X = P_X - P_X = 0$$

$$P_X \cdot M_X = P_X (I - P_X) = P_X \cdot I - P_X P_X = P_X - P_X = 0$$