Ana I, Blatt 2

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1) a) <u>Beh</u>: $\sum_{k=1}^{n} k^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \quad (\forall n \in \mathbb{N}_0)$

<u>Bew</u>: <u>$n=0$</u>: l.S. = r.S. = 0 ✓

<u>$n \to n+1$</u>: $\cancel{\sum_{k=1}^{n+1} k^3 \overset{(I.V.)}{=}}$ mit r.S. starten!

$$\frac{1}{4}(n+1)^4 + \frac{1}{2}(n+1)^3 + \frac{1}{4}(n+1)^2$$

$$= \frac{1}{4}(n^4 + 4n^3 + 6n^2 + 4n + 1) + \frac{1}{2}(n^3 + 3n^2 + 3n + 1) + \frac{1}{4}(n^2 + 2n + 1)$$

$$= \frac{1}{4}n^4 + \left(1 + \frac{1}{2}\right)n^3 + \left(3 + \frac{1}{4}\right)n^2 + 3n + 1$$

$$\overset{I.V.}{=} \sum_{k=1}^{n} k^3 + \cancel{k^3}(n+1)^3 = \sum_{k=1}^{n+1} k^3 \qquad \blacksquare$$

b) <u>Beh</u>: $\prod_{k=1}^{n-1}\left(1 + \frac{1}{k}\right)^k = \frac{n^n}{n!} \quad (\forall n \in \mathbb{N})$

<u>Bew</u>: <u>$n=1$</u>: l.S. = 1 („leeres Produkt"!)

r.S. = 1 ✓

<u>$n \to n+1$</u>: $\prod_{k=1}^{n}\left(1 + \frac{1}{k}\right)^k \overset{I.V.}{=} \frac{n^n}{n!}\left(1 + \frac{1}{n}\right)^n$

$$= \frac{n^n}{n!} \cdot \frac{(n+1)^n}{n^n} = \frac{(n+1)^{n+1}}{(n+1)!} \qquad \blacksquare$$

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2) a) Beh: $d: \mathbb{R}\times\mathbb{R} \to \mathbb{R}, \quad (x,y) \mapsto |x-y|$ ist eine Metrik auf $\mathbb{R}$.

Bew: (i) $d(x,x) = |x-x| = |0| = 0$ ✓

Für $x \neq y$ ist $d(x,y) = |x-y| \neq 0$, da $x-y \neq 0$.

(ii) $d(x,y) = |x-y| = |-(x-y)| = |y-x| = d(y,x)$

(iii) $d(x,y) + d(y,z) = |x-y| + |y-z|$

$\geq |(x-y) + (y-z)| = |x-z| = d(x,y)$

Für $\alpha, \beta \in \mathbb{R}$ gilt: $|\alpha| + |\beta| \geq |\alpha + \beta|$.

Bew: 1. Fall: $\alpha+\beta \geq 0 \Rightarrow |\alpha|+|\beta| \geq \alpha+\beta = |\alpha+\beta|$

2. Fall: $\alpha+\beta < 0 \Rightarrow |\alpha|+|\beta| \geq -\alpha-\beta = |\alpha+\beta|$

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b) Beh: $|x-y| \geq ||x|-|y|| \quad (\forall x,y \in \mathbb{R})$

~~Bew: Fall 1: $0 \leq x, y$~~

~~$\Rightarrow ||x|-|y|| = |x-y| \leq |x-y|$ ✓~~

~~Fall 2: $x, y \leq 0$~~

~~$\Rightarrow ||x|-|y|| = |-x-(-y)| = |x-y| \leq |x-y|$ ✓~~

~~Fall 3: $x \leq 0 \leq y$~~

~~$\Rightarrow ||x|-|y|| = |-x-y|$~~

Bew: $x = x-y+y \Rightarrow |x| \leq |x-y| + |y|$

$\Rightarrow |x| - |y| \leq |x-y| \quad (1)$

$y = y-x+x \Rightarrow |y| \leq |y-x| + |x|$

$\Rightarrow |y| - |x| \leq |y-x| \Rightarrow -(|x|-|y|) \leq |x-y| \quad (2)$

$(1) \& (2) \Rightarrow ||x|-|y|| \leq |x-y|$ ☒