Labour Economics
Solutions to Problem Set 6

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Introduction & Motivation

A **Canonical SBTC model** (Goldin and Katz 2008; Katz and Murphy 1992; Tinbergen 1974)
Technology shifts *skill demand* ⇒ college premium changes

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B **Ricardian task model** (Acemoglu and Autor 2011; Autor, Levy, and Murnane 2003)
Technology shifts *task costs* ⇒ reallocation and polarization

$\bigvee$

C **Decomposition approaches** (Card, Heining, and Kline 2013; Song et al. 2019)
Variance via *worker effects, firm premia, sorting*, residuals

Canonical Model: Setup and Approach

The canonical Skill-Biased Technological Change (SBTC) model explains wage changes through technology-driven shifts in skill use

$$\underbrace{Y}_{\text{output}} = \left[\underbrace{(A_L L)^{\frac{\sigma-1}{\sigma}}}_{\substack{\text{low-skill} \\ \text{effective input}}} + \underbrace{(A_H H)^{\frac{\sigma-1}{\sigma}}}_{\substack{\text{high-skill} \\ \text{effective input}}} \right]^{\frac{\sigma}{\sigma-1}}$$

$$\underbrace{\log\left(\frac{w_t^H}{w_t^L}\right)}_{\text{high skill premium}} = \frac{\sigma-1}{\sigma} \underbrace{\log\left(\frac{A_t^H}{A_t^L}\right)}_{\substack{\text{relative technology} \\ \text{(SBTC)}}} - \frac{1}{\sigma} \underbrace{\log\left(\frac{H_t}{L_t}\right)}_{\substack{\text{relative} \\ \text{skill supply}}}$$

Task 1: Parameters of Interest

$$y_t = \left[(\theta_{ht} H_t)^{\frac{\sigma-1}{\sigma}} + (\theta_{ct} C_t)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

$$\log\left(\frac{w_t^c}{w_t^h}\right) = \frac{\sigma-1}{\sigma} D_t - \frac{1}{\sigma}\log\left(\frac{C_t}{H_t}\right)$$

- **Elasticity of substitution (σ):** governs how strongly relative wages respond to relative supply. $\sigma \geq 0; \quad \sigma > 1$: substitutes, $\uparrow \sigma \Rightarrow \uparrow$ substitutability
- **Relative "demand shifter" (D_t):** captures weighted relative technology for college vs. high-school labor over time. In the CES setup it is:

$$D_t = \log\left(\frac{\theta_{ct}}{\theta_{ht}}\right),$$

i.e. increases in θ_{ct}/θ_{ht} shift demand toward college labour and raise the college premium.

Task 2: Parameter Identification

$$y_t = \left[(\theta_{ht} H_t)^{\frac{\sigma-1}{\sigma}} + (\theta_{ct} C_t)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

$$\log\left(\frac{w_t^c}{w_t^h}\right) = \frac{\sigma-1}{\sigma} D_t - \frac{1}{\sigma}\log\left(\frac{C_t}{H_t}\right)$$

σ and D_t are **not** separately identified as D_t is an unobserved time-varying intercept:

$$D_t \equiv log\left(\frac{w_t^c}{w_t^h}\right) + \frac{1}{\sigma} log\left(\frac{C_t}{H_t}\right) \quad \text{for any } \sigma$$

We need additional structure, a common approach is to put **parametric structure on** D_t:

$$D_t = \gamma_0 + \gamma_1 t \quad \Rightarrow$$

$$\log\left(\frac{w_t^c}{w_t^h}\right) = \frac{\sigma-1}{\sigma}\gamma_0 + \frac{\sigma-1}{\sigma}\gamma_1 t - \frac{1}{\sigma}\log\left(\frac{C_t}{H_t}\right)$$

Task 3: Elasticity and Substitution

$$y_t = \left[(\theta_{ht} H_t)^{\frac{\sigma-1}{\sigma}} + (\theta_{ct} C_t)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

$$\log\left(\frac{w_t^c}{w_t^h}\right) = \frac{\sigma-1}{\sigma} D_t - \frac{1}{\sigma}\log\left(\frac{C_t}{H_t}\right)$$

$$\frac{1}{\sigma} \approx 0.5 \text{ to } 0.66 \quad \Longrightarrow$$

$$\sigma \approx \frac{1}{0.66} \text{ to } \frac{1}{0.5} \approx 1.5 \text{ to } 2$$

Since $\sigma > 1$,

- C and H are substitutes (imperfect substitutes, as σ is finite)
- Observed increases in the college wage premium must be attributed to increases in D_t

Task 4: Drivers of the College Premium

$$y_t = \left[(\theta_{ht} H_t)^{\frac{\sigma-1}{\sigma}} + (\theta_{ct} C_t)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

$$\log\left(\frac{w_t^c}{w_t^h}\right) = \frac{\sigma-1}{\sigma} D_t - \frac{1}{\sigma}\log\left(\frac{C_t}{H_t}\right)$$

Empirical studies find $\sigma \approx 1.5$ to 2
$\Rightarrow \frac{1}{\sigma} \approx 0.5$ to 0.66
$\frac{\sigma-1}{1} \approx 0.33$ to 0.5

- $\uparrow \log(C_t/H_t)$ (data) $\Rightarrow \downarrow$ wage premium
- $\uparrow \log(w_t^c/w_t^h)$ (data) $\Rightarrow \Delta D_t > \frac{1}{\sigma}\Delta\log(C_t/H_t)$

Relative demand / productivity shifts dominated relative supply effects over the last 30 years

Task 5: Model Limitations

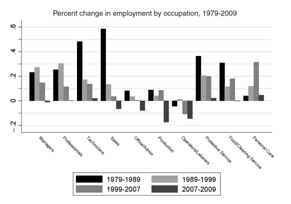
$$y_t = \left[(\theta_{ht} H_t)^{\frac{\sigma-1}{\sigma}} + (\theta_{ct} C_t)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

$$\log\left(\frac{w_t^c}{w_t^h}\right) = \frac{\sigma-1}{\sigma} D_t - \frac{1}{\sigma} \log\left(\frac{C_t}{H_t}\right)$$

(a) Conflation of skills and tasks

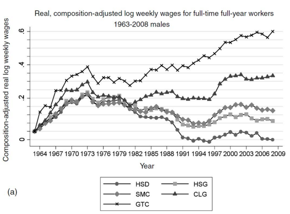
(b) Linear trend valid pre-1995

(c) Job Polarization



(d) Wage Polarization

(e) Declining Wage Predictions



Ricardian Model: Setup and Approach

- Task-based model: allows for a distinction between tasks and skills
- Used to explain how technology affects skill demand, earnings, and the assignment of skills to tasks

$$\underbrace{Y}_{\text{output}} = exp\left[\int_0^1 ln \underbrace{y(i)}_{\substack{\text{production}\\\text{level}\\\text{of task i}}} di\right]$$

Task 1: Task Production

$$lnA_M - lnA_H + \beta_H(I_H) + lnM - lnH - ln(I_H - I_L) + ln(1 - I_H) = 0$$

$$lnA_L - lnA_M + \beta_L(I_L) + lnL - lnM + ln(I_H - I_L) - ln(I_L) = 0$$

$$\underbrace{y(i)}_{\text{output}} = \underbrace{A_L}_{\text{factor-augmenting technology}} \alpha_L(i) l(i) + A_M \underbrace{\alpha_M(i)}_{\text{productivity of medium skill type in task } i} m(i) + A_H \alpha_H(i) \underbrace{h(i)}_{\text{number of high skill workers allocated to task } i}$$

Task 2: Skills vs. Tasks

$$lnA_M - lnA_H + \beta_H(I_H) + lnM - lnH - ln(I_H - I_L) + ln(1 - I_H) = 0$$

$$lnA_L - lnA_M + \beta_L(I_L) + lnL - lnM + ln(I_H - I_L) - ln(I_L) = 0$$

- Skill: worker's endowment of capabilities to perform various tasks
- Task: unit of work activity that produces output

Why is the distinction important? What is the difference compared to the canonical model?

Task 3: Comparative Advantage

$$lnA_M - lnA_H + \beta_H(I_H) + lnM - lnH - ln(I_H - I_L) + ln(1 - I_H) = 0$$

$$lnA_L - lnA_M + \beta_L(I_L) + lnL - lnM + ln(I_H - I_L) - ln(I_L) = 0$$

Assumption: Both fractions are continuous, differentiable, and strictly decreasing in i

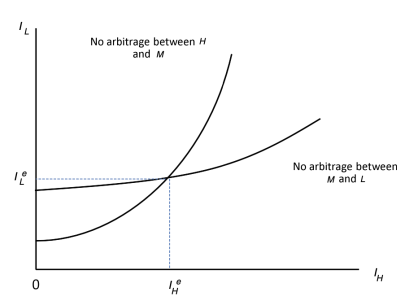
$$\frac{\alpha_L(i)}{\alpha_M(i)} \quad \text{and} \quad \frac{\alpha_M(i)}{\alpha_H(i)}$$

So higher indices correspond to more complex tasks, in which higher-skilled workers perform better than lower skilled workers (bigger comparative advantage; clean "sorting")

Task 4: Equilibrium Allocation

$$lnA_M - lnA_H + \beta_H(I_H) + lnM - lnH - ln(I_H - I_L) + ln(1 - I_H) = 0$$

$$lnA_L - lnA_M + \beta_L(I_L) + lnL - lnM + ln(I_H - I_L) - ln(I_L) = 0$$



Determination of equlibrium threshold tasks

Law of one price

$$w_L = p(i)A_L\alpha_L(i) \quad \text{for any } i < I_L$$
$$w_M = p(i)A_M\alpha_M(i) \quad \text{for any } I_L < i < I_H$$
$$w_H = p(i)A_H\alpha_H(i) \quad \text{for any } i > I_H$$

Task 5: AI and Skills (comparative statics)

$$A_H \uparrow \Rightarrow I_H \downarrow \Rightarrow I_L \downarrow$$

$$lnA_M - lnA_H + \beta_H(I_H) + lnM - lnH - ln(I_H - I_L) + ln(1 - I_H) = 0$$

$$lnA_L - lnA_M + \beta_L(I_L) + lnL - lnM + ln(I_H - I_L) - ln(I_L) = 0$$

1. "No arbitrage between H and M" - curve will shift inwards
   $\Rightarrow I_H$ decreases
2. If I_L remained constant following the downward movement of I_H, this would imply an excess of medium skill workers
   $\Rightarrow I_L$ decreases

Because the technological change is complementing and not (task-)replacing, the relative wages of high-skilled workers increase

Task 6: Task Automation (comparative statics)

$$lnA_M - lnA_H + \beta_H(I_H) + lnM - lnH - ln(I_H - I_L) + ln(1 - I_H) = 0$$

$$lnA_L - lnA_M + \beta_L(I_L) + lnL - lnM + ln(I_H - I_L) - ln(I_L) = 0$$

+ 1 or 2 additional arbitrage conditions

- We expand the production function and add an additional term: $A_K \alpha_K(i) k(i)$
- The new thresholds determining the equilibrium are now: $0 < I_L^k < I_H^k < I_K^k < 1$ (some tasks shift from H to AI, other boundaries adjust because of reallocation)
- Relative wages compress: AI replaces top tasks, triggering downward reallocation of high- and medium-skilled workers, so all skill premia fall:
  $w_H/w_M \downarrow, \quad w_M/w_L \downarrow, \quad w_H/w_L \downarrow$

Task 7: Ricardian Model – Beyond SBTC

$$lnA_M - lnA_H + \beta_H(I_H) + lnM - lnH - ln(I_H - I_L) + ln(1 - I_H) = 0$$

$$lnA_L - lnA_M + \beta_L(I_L) + lnL - lnM + ln(I_H - I_L) - ln(I_L) = 0$$

$+$ 1 or 2 additional arbitrage conditions

- Allows for a distinction between tasks and skills (can explain how same skill group reallocates across tasks when technology changes)
- Incorporates comparative advantage and explains how technology affects skill demand, earnings, and the assignment of skills to tasks
- To explain wage and job polarization, the Ricardian model allows for at least three skill groups

Decomposition Approaches: Setup and Approach

Remember: Decomposing the sources of outcome variation separates signals from each other (and noise), e.g. Regression ANOVA

$$
\begin{aligned}
Var(Y_i) &= Var(\widehat{Y}_i) + Var(e_i) \\
&= Var(\alpha + \hat{\beta} X_i) + Var(e_i) \\
&= Var(\hat{\beta} X_i) + Var(e_i) \\
&= \hat{\beta}^2 Var(X_i) + Var(e_i)
\end{aligned}
$$

Task 1: Card et al. (2013) – AKM-Decomposition

$$w_{it} = \alpha_i + \psi_{J(i,t)} + X_{it}'\beta + \varepsilon_{it} \quad \text{(AKM model)}$$

$$\begin{aligned} Var(w_{it}) = & \underbrace{Var(\alpha_i)}_{\text{worker diff}} + \underbrace{Var(\psi_{J(i,t)})}_{\text{firm diff}} + \underbrace{Var(X_{it}'\beta)}_{\text{observables}} + \underbrace{Var(\varepsilon_{it})}_{\text{residual}} \\ & + 2 \underbrace{Cov(\alpha_i, \psi_{J(i,t)})}_{\text{assortative matching / sorting}} + 2 \underbrace{Cov(\alpha_i, X_{it}'\beta)}_{\text{ability–observables correlation}} \\ & + 2 \underbrace{Cov(\psi_{J(i,t)}, X_{it}'\beta)}_{\text{firm premia–observables correlation}} + \underbrace{\text{covariances involving } \varepsilon_{it}}_{\text{0 (by construction)}} \end{aligned}$$

- Worker effects: approx. 40%
- Firm effects: approx. $20 - 25\%$
- Assortative matching: approx. 33%

Task 2: Song et al. (2019) – Between-Within-Decomposition

$$w_{iJ} \equiv w_J + e_{iJ} \quad \text{(firm average with individual deviation)}$$

$$\begin{aligned}
Var(w_{iJ}) &= Var(w_J + e_{iJ}) \\
&= Var(w_J) + Var(e_{iJ}) + 2Cov(w_J, e_{iJ}) \\
&= Var(w_J) + Var(e_{iJ}) + 2\left(\mathbb{E}[w_J e_{iJ}] - \mathbb{E}[w_J]\mathbb{E}[e_{iJ}]\right) \\
&= Var(w_J) + Var(e_{iJ}) + 2\left(\mathbb{E}[\mathbb{E}[w_J e_{iJ} \mid J]] - \mathbb{E}[w_J]\mathbb{E}[e_{iJ}]\right) \\
&= Var(w_J) + Var(e_{iJ}) + 2\left(\mathbb{E}[w_J \mathbb{E}[e_{iJ}]] - \mathbb{E}[w_J]\mathbb{E}[e_{iJ}]\right) \\
&= Var(w_J) + Var(e_{iJ}) + 2\left(\mathbb{E}[w_J * 0] - \mathbb{E}[w_J] * 0\right) \\
&= \underbrace{Var(w_J)}_{\text{Between-firm Component}} + \underbrace{Var(e_{iJ})}_{\text{Within-firm Component}}
\end{aligned}$$

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